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A continuum limit of the relativistic Toda lattice: asymptotic theory of discrete Laurent orthogonal polynomials with varying recurrence coefficients

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Abstract

We consider the continuum limit of the relativistic Toda lattice. In particular, we propose a method in order to 'integrate' this system of nonlinear partial differential equations for some particular initial data and boundary conditions, before possible shocks. First, we recall the relation between the finite relativistic Toda lattice and the theory of discrete Laurent orthogonal polynomials. Our analysis is then based on some results for the asymptotic theory of discrete Laurent orthogonal polynomials with varying recurrence coefficients and the connection with a constrained and weighted extremal problem for logarithmic potentials.

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1. Introduction

1.1. The finite relativistic Toda lattice

After some change of variables Bruschi and Ragnisco [4, 5] showed that the finite relativistic Toda lattice (RTL), introduced by Ruijsenaars [30], can be written in the form

$$\begin{aligned} \dot{a}_{n,N} &= a_{n,N}(b_{n-1,N} - b_{n,N}), && 1 \leqslant n \leqslant N, \\ \dot{b}_{n,N} &= b_{n,N}(a_{n,N} - a_{n+1,N} + b_{n-1,N} - b_{n+1,N}), && 1 \leqslant n \leqslant N-1, \end{aligned} \tag{1.1}$$

with operator data $a_{n,N} > 0, 1 \leqslant n \leqslant N$ and $b_{n,N} > 0, 1 \leqslant n \leqslant N-1$, and with $b_{0,N} \equiv 0, b_{N,N} \equiv 0$. This system of nonlinear differential equations was already studied in [4, 5, 8, 9, 29, 33–36]; for a review see [24]. There exist several matrix representations for system (1.1). For example, using the two bidiagonal matrices

$$L_N = \begin{pmatrix} a_{1,N} & 1 & 0 & \cdots & 0 \\ 0 & a_{2,N} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & a_{N,N} \end{pmatrix},$$

$$M_N = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ -b_{1,N} & 1 & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 & 0 \\ 0 & \cdots & 0 & -b_{N-1,N} & 1 \end{pmatrix}, \tag{1.2}$$

see [35, 36] and also [9], system (1.1) has the Lax form

$$\dot{L}_N = L_N A_N - B_N L_N \qquad \dot{M}_N = M_N A_N - B_N M_N, \tag{1.3}$$

where $A_N = -\left(M_N^{-1} L_N\right)_-$ and $B_N = -\left(L_N M_N^{-1}\right)_-$, proving integrability. Here we use X_- to denote the strictly lower triangular part of X.

In a similar way as was done by Moser [27, 28] for the nonrelativistic case, in [9] the authors solved the corresponding Cauchy problem for positive initial operator data, which consists of finding $a_{n,N}(t), b_{n,N}(t)$, with $t > 0$, satisfying the differential equations (1.1). Here they made use of a bijective spectral transform of the form

$$\begin{Bmatrix} a_{1,N}, \ldots, a_{n,N} > 0 \\ b_{1,N}, \ldots, b_{N-1,N} > 0 \end{Bmatrix} \longmapsto \begin{Bmatrix} 0 < \lambda_{1,N} < \cdots < \lambda_{N,N} \\ w_{1,N}, \ldots, w_{N,N} > 0, \sum_{j=1}^{N} w_{j,N} = 1 \end{Bmatrix}, \tag{1.4}$$

based on the spectral theory of the pair of bidiagonal matrices L_N and M_N, and then computed the exact evolution of the spectral data.

Direct spectral transform. Start from $a_{n,N} > 0, 1 \leqslant n \leqslant N$ and $b_{n,N} > 0, 1 \leqslant n \leqslant N-1$, and

- construct the pair of bidiagonal matrices (L_N, M_N) as in (1.2),
- compute their positive and simple generalized eigenvalues $\lambda_{1,N} < \cdots < \lambda_{N,N}$ and the corresponding left and right eigenvectors $\vec{u}_{j,N}$ and $\vec{v}_{j,N}$, normalized to have their first component equal to 1:

$$L_N \vec{v}_{j,N} = \lambda_{j,N} M_N \vec{v}_{j,N}, \qquad \vec{u}_{j,N}^{\mathrm{T}} L_N = \lambda_{j,N} \vec{u}_{j,N}^{\mathrm{T}} M_N, \qquad 1 \leqslant j \leqslant N,$$

- define $w_{j,N} = \left(\vec{u}_{j,N}^{\mathrm{T}} M_N \vec{v}_{j,N}\right)^{-1} > 0, 1 \leqslant j \leqslant N$, which satisfy $\sum_{j=1}^{N} w_{j,N} = 1$.

Evolution of the spectral data. The $\lambda_{j,N}$ are time independent and

$$w_{j,N}(t) = \frac{w_{j,N}(0)\, \mathrm{e}^{-\lambda_{j,N} t}}{\sum_{k=1}^{N} w_{k,N}(0)\, \mathrm{e}^{-\lambda_{k,N} t}}, \qquad 1 \leqslant j \leqslant N. \tag{1.5}$$

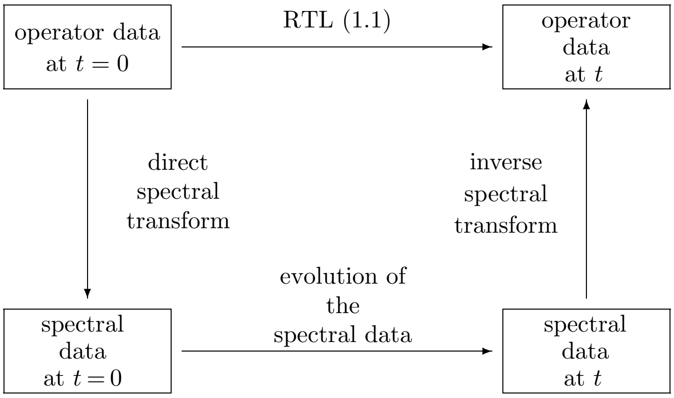

Figure 1. The scheme of the direct and inverse spectral problem.

Inverse spectral transform. Start from positive $\lambda_{j,N}, w_{j,N}, 1 \leqslant j \leqslant N$, satisfying $\sum_{j=1}^{N} w_{j,N} = 1$, and compute the terminating continued T-fraction [13, 15]

$$\sum_{j=1}^{N} \frac{w_{j,N}}{z - \lambda_{j,N}} = \cfrac{1}{z - a_{1,N} - \cfrac{b_{1,N}z}{z - a_{2,N} - \cfrac{b_{2,N}z}{z - a_{3,N} - \ddots - \cfrac{b_{N-1,N}z}{z - a_{N,N}}}}}.$$

For a proof of the properties of the spectral data, their evolution in time and the inverse spectral transform, we refer to [9]. The scheme in figure 1 then illustrates how to solve the finite RTL.

In [9] the authors also showed that this spectral transform is closely connected to the theory of Laurent orthogonal polynomials [13, 15–18, 31]. Note that the connection of the RTL with such polynomials was mentioned in the literature before, see, e.g., [38]. In particular we can define monic polynomials $P_{n,N}$ by

$$P_{0,N} \equiv 1, \qquad P_{n,N}(z) = \det(zM_{n,N} - L_{n,N}), \qquad 1 \leqslant n \leqslant N, \tag{1.6}$$

where $L_{n,N}$ and $M_{n,N}$ are the $n \times n$ upper left corner blocks of L_N and M_N, respectively. Evidently, the $\lambda_{j,N}$ are now the zeros of $P_{N,N}$. An equivalent definition of these polynomials is given by the recurrence relation

$$P_{n,N}(z) = (z - a_{n,N})P_{n-1,N}(z) - b_{n-1,N}zP_{n-2,N}(z), \qquad 1 \leqslant n \leqslant N, \tag{1.7}$$

with $P_{0,N} \equiv 1$, $P_{-1,N} \equiv 0$, emphasizing the relation with the generalized eigenvalue problem for the pair of matrices (L_N, M_N). It is then easily proven that

$$\sum_{j=1}^{N} P_{n,N}(\lambda_{j,N})P_{m,N}(\lambda_{j,N})\frac{w_{j,N}}{(\lambda_{j,N})^n} = 0, \qquad \text{if} \quad 0 \leqslant m < n \leqslant N - 1,$$

so that they are the monic Laurent orthogonal polynomials with respect to the discrete probability measure $\mu_N = \sum_{j=1}^{N} w_{j,N}\delta_{\lambda_{j,N}}$. Here by $\delta_{\lambda_{j,N}}$ we denote the Dirac measure at the point $\lambda_{j,N}$.

1.2. The continuum limit of the relativistic Toda lattice

We now look what happens with the solution of system (1.1), with $t \to Nt$, for large N. In particular, assume that the limits

$$\lim_{n/N\to x} a_{n,N}(Nt) = a(x,t), \qquad 0 \leqslant x \leqslant 1, \tag{1.8}$$

$$\lim_{n/N\to x} b_{n,N}(Nt) = b(x,t), \qquad 0 \leqslant x \leqslant 1, \qquad b(0,t) = b(1,t) = 0, \tag{1.9}$$

exist (which is then uniform in x for $N \to \infty$), where the limit is taken over any sequence $\{(n_j, N_j)\}_{j\geqslant 1}$ for which $n_j \to \infty$, $N_j \to \infty$ and $n_j/N_j \to x$ as $j \to \infty$. Note that these limit functions are continuous and non-negative. Under appropriate conditions, $a(x,t)$ and $b(x,t)$ then satisfy the system of nonlinear partial differential equations

$$\frac{\partial a}{\partial t} = -a\frac{\partial b}{\partial x} \qquad \frac{\partial b}{\partial t} = -b\left(\frac{\partial a}{\partial x} + 2\frac{\partial b}{\partial x}\right), \tag{1.10}$$

with non-negative initial conditions $a(x,0), b(x,0), 0 \leqslant x \leqslant 1$ and boundary conditions $b(0,t) \equiv 0, b(1,t) \equiv 0$. We call this system the continuum limit of the relativistic Toda lattice. It is easy to see that, for positive initial conditions, this is an example of a hyperbolic PDE. Note that, applying the bijective transformation

$$a, b \geqslant 0 \longmapsto (\alpha, \beta) = \left(\left(\sqrt{a+b} - \sqrt{b}\right)^2, \left(\sqrt{a+b} + \sqrt{b}\right)^2\right) \quad \text{with} \quad 0 \leqslant \alpha \leqslant \beta, \tag{1.11}$$

system (1.10) reduces to the simpler form

$$\frac{\partial \alpha}{\partial t} = \frac{\sqrt{\alpha}(\sqrt{\beta} - \sqrt{\alpha})}{2}\frac{\partial \alpha}{\partial x} \qquad \frac{\partial \beta}{\partial t} = -\frac{\sqrt{\beta}(\sqrt{\beta} - \sqrt{\alpha})}{2}\frac{\partial \beta}{\partial x}. \tag{1.12}$$

Taking $a \equiv 0$, or equivalently $\alpha \equiv 0$, the system reduces to the well-known inviscid Burgers equation.

In this paper we first of all study the asymptotic zero distribution of Laurent orthogonal polynomials with varying recurrence coefficients, see section 2. Next, we try to solve the continuum limit of the RTL (1.10), with some particular initial conditions $a(x,0), b(x,0), 0 \leqslant x \leqslant 1$, and with boundary conditions $b(0,t) \equiv 0, b(1,t) \equiv 0$, before possible shocks. In particular, in sections 3, 4 and 5 we give some justification (under rather strong conditions) for the procedure below, based on the integration of (1.1) as in section 1.1 and the asymptotic theory of discrete Laurent orthogonal polynomials. After constructing initial discrete operator data for system (1.1) from $a(\cdot,0), b(\cdot,0)$, we assume that for $t > 0$ the limits (1.8), (1.9) exist and satisfy (1.10). To justify the existence of these limits we need an asymptotic formula for the ratio of two consecutive discrete Laurent orthogonal polynomials. However, as in the case of discrete orthogonal polynomials, for discrete Laurent orthogonal polynomials this is still an open problem.

- Start from initial operator data $a(\cdot,0) = a(\cdot)$ and $b(\cdot,0) = b(\cdot)$ in $C[0,1]$ for which

  $$a(x) > 0, \qquad 0 \leqslant x \leqslant 1, \qquad b(x) > 0, \qquad 0 < x < 1, \quad b(0) = b(1) = 0.$$

  Furthermore, we require that $\log b(x) = o\left(\frac{1}{x(1-x)}\right)$ for $x \downarrow 0, x \uparrow 1$, and that for each $\lambda \in [\min_{0\leqslant x\leqslant 1} \alpha(x), \max_{0\leqslant x\leqslant 1} \beta(x)]$ the set

  $$[x_-(\lambda), x_+(\lambda)] := \{x \in [0,1] : \lambda \in [\alpha(x), \beta(x)]\}$$

  is an interval, where the functions α, β are defined from a and b by (1.11). The spectral data at time $t = 0$ are then

$$\sigma(0) = \int_0^1 \upsilon_{[\alpha(x),\beta(x)]}\,\mathrm{d}x, \tag{1.13}$$

$$R(\lambda, 0) = -\int_0^{x_-(\lambda)} (g_{[\alpha(u),\beta(u)]}(\lambda,\infty) + g_{[\alpha(u),\beta(u)]}(\lambda, 0))\,\mathrm{d}u, \tag{1.14}$$

where, for $0 < \alpha < \beta$,

$$\frac{\mathrm{d}\upsilon_{[\alpha,\beta]}(\lambda)}{\mathrm{d}\lambda} = \begin{cases} \dfrac{1}{2\pi}\dfrac{\lambda + \sqrt{\alpha\beta}}{\lambda\sqrt{(\beta-\lambda)(\lambda-\alpha)}}, & \lambda \in [\alpha,\beta], \\ 0, & \text{elsewhere,} \end{cases}$$

and

$$g_{[\alpha,\beta]}(z,\infty) = \begin{cases} \log\left|\dfrac{2z - (\alpha+\beta) + 2\sqrt{(z-\alpha)(z-\beta)}}{\beta-\alpha}\right|, & z \in \mathbb{C}\backslash[\alpha,\beta], \\ 0, & z \in [\alpha,\beta], \end{cases}$$

$$g_{[\alpha,\beta]}(z,0) = \begin{cases} \log\left|\dfrac{(\alpha+\beta)z - 2\alpha\beta + 2\sqrt{\alpha\beta}\sqrt{(z-\alpha)(z-\beta)}}{z(\beta-\alpha)}\right|, & z \in \mathbb{C}\backslash[\alpha,\beta], \\ 0, & z \in [\alpha,\beta], \end{cases}$$

which are the Green functions of the complement of $[\alpha, \beta]$ with pole at ∞ and 0, respectively.

- Take, for $t > 0$, the time evolution

$$\sigma(t) = \sigma(0) = \sigma, \tag{1.15}$$

$$R(\lambda, t) = -\lambda t + R(\lambda, 0) - \max_{\lambda\in\mathrm{supp}(\sigma)} (-\lambda t + R(\lambda, 0)), \tag{1.16}$$

with $\mathrm{supp}(\sigma) = [\min_{0\leqslant x\leqslant 1} \alpha(x), \max_{0\leqslant x\leqslant 1} \beta(x)]$.

- Finally, consider the logarithmic extremal problem with constraint σ and external field $-\frac{1}{2}(xU^{\delta_0} + R)$, linearly depending on the mass x of the measure:

$$J_R(\tau_x; x) = \min_{\mu(\mathbb{R})=x,\mu\leqslant\sigma} J_R(\mu; x), \qquad 0 < x < 1,$$

with

$$J_R(\mu; x) := \iint \log\frac{1}{|y-y'|}\,\mathrm{d}\mu(y)\,\mathrm{d}\mu(y') + \int (x\log y - R(y))\,\mathrm{d}\mu(y).$$

These measures τ_x, $0 < x < 1$, are also characterized as the unique measures with mass x, constrained by σ and satisfying the variational conditions

$$U^{\tau_x}(\lambda) + \frac{x}{2}\log\lambda - \frac{1}{2}R(\lambda) \begin{cases} \geqslant w(\tau_x), & \lambda \in \mathrm{supp}(\sigma - \tau_x), \\ = w(\tau_x), & \lambda \in \mathrm{supp}(\sigma-\tau_x) \cap \mathrm{supp}(\tau_x) =: \Sigma(\tau_x), \\ \leqslant w(\tau_x), & \lambda \in \mathrm{supp}(\tau_x), \end{cases}$$

for some constants $w(\tau_x) \in \mathbb{R}$, where $U^\mu(z) = -\int \log|z-y|\,\mathrm{d}\mu(y)$. As long as, for $t > 0$, the regions of equilibrium $\Sigma(\tau_x, t)$, $0 < x < 1$, are all intervals (different from a singleton), we have

$$\Sigma(\tau_x, t) = [\alpha(x,t), \beta(x,t)], \qquad 0 < x < 1. \tag{1.17}$$

After applying the inverse of (1.11), this then gives the solution of (1.10) at $t > 0$.

For the continuum limit of the Toda lattice a similar analysis was done in [1] by means of the asymptotic theory of discrete orthogonal polynomials, see, e.g., [10, 19, 20] for a more detailed study. The relation with extremal problems occurred in the literature before, see, e.g., the zero dispersion limit of Korteweg–de Vries equation [26] and the defocusing cubic nonlinear Schrödinger equation [14].

2. The asymptotic zero distribution of Laurent orthogonal polynomials with varying recurrence coefficients

2.1. Introduction

Let μ be a positive measure on $\mathbb{R}^+$ for which all the strong moments exist. Then, the sequence of monic Laurent orthogonal polynomials $\{P_n\}_{n\geqslant 0}$ with respect to μ are defined by the orthogonality relations

$$\int P_n(y)y^k \frac{\mathrm{d}\mu(y)}{y^n} = 0, \qquad k = 0, \ldots, n-1. \tag{2.1}$$

Such polynomials P_n have n positive and simple zeros (see, e.g., [9, 31]) and we can associate with each polynomial P_n the normalized zero counting measure

$$\nu(P_n) := \frac{1}{n} \sum_{P_n(z)=0} \delta_z, \tag{2.2}$$

where δ_z denotes the Dirac point mass at z. If $\nu(P_n) \to \nu$ in the sense of weak-$\star$ convergence, which means that

$$\lim_{n\to\infty} \int f\,\mathrm{d}\nu(P_n) = \int f\,\mathrm{d}\nu$$

for every bounded and continuous function f on $\mathbb{R}^+$, then we call the probability measure ν the *asymptotic zero distribution* of the sequence $\{P_n\}_{n\geqslant 0}$.

Monic Laurent orthogonal polynomials with respect to a probability measure on $\mathbb{R}^+$ satisfy a three-term recurrence relation of the form

$$P_n(z) = (z - a_n)P_{n-1}(z) - b_{n-1}zP_{n-2}(z), \qquad n \geqslant 1, \tag{2.3}$$

with $a_n > 0, b_n > 0$ and initial conditions $P_0 \equiv 1$, $P_{-1} \equiv 0$. In [15, 31] a Favard theorem was proven, indicating that any sequence of polynomials which satisfies a recurrence of the kind (2.3) is a sequence of monic Laurent orthogonal polynomials with respect to a probability measure on $\mathbb{R}^+$. In the introduction we mentioned a finite version of this theorem, used in [9] to solve the finite RTL. So, it is clear that Laurent orthogonal polynomials with respect to a positive measure on the positive real line can be studied from their recurrence relation. In [31] the authors then obtained the following result for the asymptotic zero distribution.

Theorem 2.1. *If for the recurrence coefficients in (2.3) we have the limits*

$$\lim_{n\to\infty} a_n = a > 0, \qquad \lim_{n\to\infty} b_n = b > 0,$$

then the asymptotic zero distribution of the (Laurent orthogonal) polynomials P_n is given by the probability measure $\upsilon_{[\alpha,\beta]}$ with density

$$\frac{\mathrm{d}\upsilon_{[\alpha,\beta]}(y)}{\mathrm{d}y} = \begin{cases} \dfrac{1}{2\pi} \dfrac{y + \sqrt{\alpha\beta}}{y\sqrt{(\beta - y)(y - \alpha)}}, & y \in [\alpha, \beta], \\ 0, & \textit{elsewhere}, \end{cases} \tag{2.4}$$

where $\alpha = (\sqrt{a+b} - \sqrt{b})^2$ and $\beta = (\sqrt{a+b} + \sqrt{b})^2, 0 < \alpha < \beta$.

Remark 2.2. Let $0 < \alpha < \beta$. The asymptotic zero distribution (2.4) is equal to the convex combination $\upsilon_{[\alpha,\beta]} = \frac{1}{2}\omega_{[\alpha,\beta]} + \frac{1}{2}\hat{\delta}_{0,[\alpha,\beta]}$, where

$$\frac{\mathrm{d}\omega_{[\alpha,\beta]}(y)}{\mathrm{d}y} = \begin{cases} \dfrac{1}{\pi\sqrt{(\beta-y)(y-\alpha)}}, & y \in [\alpha,\beta], \\ 0, & \text{elsewhere}, \end{cases}$$

is the arcsine measure on $[\alpha, \beta]$ and

$$\frac{\mathrm{d}\hat{\delta}_{0,[\alpha,\beta]}(y)}{\mathrm{d}y} = \begin{cases} \dfrac{1}{\pi}\dfrac{\sqrt{\alpha\beta}}{y\sqrt{(\beta-y)(y-\alpha)}}, & y \in [\alpha,\beta], \\ 0, & \text{elsewhere}, \end{cases}$$

is the balayage of the Dirac measure δ_0 on $[\alpha, \beta]$ [32, chapter II, (4.47)]. By [32, example 4.8, p 118], it is then the unique probability measure on $[\alpha, \beta]$ minimizing the weighted logarithmic energy $\int U^\nu \,\mathrm{d}\nu + 2\int Q\,\mathrm{d}\nu$ with external field $Q(y) = \frac{1}{2}\log y$. All this is in fact closely related to a result found in the theory of orthogonal polynomials with respect to varying weights (see, e.g., [12]) since there exists a probability measure μ so that the polynomial P_n, defined by (2.3), is the monic orthogonal polynomial of degree n for the varying measure $y^{-n}\,\mathrm{d}\mu(y)$.

Our goal is to generalize theorem 2.1 to Laurent orthogonal polynomials with varying recurrence coefficients. A similar thing was done in [22] in the case of orthogonal polynomials. Let for each $N \in \mathbb{N}$ two sequences $\{a_{n,N}\}_{n\geqslant 1}$ and $\{b_{n,N}\}_{n\geqslant 1}$ of positive recurrence coefficients be given. We now study the doubly indexed sequence of polynomials $\{P_{n,N}\}$, generated by the recurrence

$$P_{n,N}(z) = (z - a_{n,N})P_{n-1,N}(z) - b_{n-1,N}zP_{n-2,N}(z), \qquad n \geqslant 1, \tag{2.5}$$

with the initial conditions $P_{0,N} \equiv 1$, $P_{-1,N} \equiv 0$. In particular we are interested in finding the asymptotic zero distributions

$$\lim_{n/N\to x} \nu(P_{n,N}), \qquad x > 0,$$

where the limit is taken over any sequence $\left\{\nu\left(P_{n_j,N_j}\right)\right\}_{j\geqslant 1}$ for which $n_j \to \infty$, $N_j \to \infty$ and $n_j/N_j \to x$ as $j \to \infty$. In this manner we find the following result.

Theorem 2.3. *Let for each $N \in \mathbb{N}$ two sequences $\{a_{n,N}\}_{n\geqslant 1}$ and $\{b_{n,N}\}_{n\geqslant 1}$ of positive recurrence coefficients be given and suppose there exist two continuous functions $a{:}(0,\infty) \to [0,\infty)$ and $b{:}(0,\infty) \to [0,\infty)$ for which*

$$\lim_{n/N\to x} a_{n,N} = a(x), \qquad \lim_{n/N\to x} b_{n,N} = b(x), \qquad x > 0. \tag{2.6}$$

Define for $x > 0$ the functions α, β from a and b by transformation (1.11). For the doubly indexed sequence of polynomials $\{P_{n,N}\}$ satisfying (2.5) we then have that

$$\lim_{n/N\to x} \nu(P_{n,N}) = \frac{1}{x}\int_0^x \upsilon_{[\alpha(u),\beta(u)]}\,\mathrm{d}u, \qquad x > 0, \tag{2.7}$$

where $\upsilon_{[\alpha,\beta]}$ is defined by (2.4) if $0 < \alpha < \beta$, by $\frac{1}{2}\omega_{[0,\beta]} + \frac{1}{2}\delta_0$ if $0 = \alpha < \beta$ and by δ_α if $0 \leqslant \alpha = \beta$. Here $\omega_{[0,\beta]}$ is the arcsine measure on $[0, \beta]$.

Remark 2.4. It is clear that, if the $a_{n,N}$ and $b_{n,N}$ do not depend on N, the functions α and β are constant. So, the result in theorem 2.1 is a special case of theorem 2.3.

Remark 2.5. The support of the measure (2.7) is given by the interval $[\inf_{0<u<x}\alpha(u), \sup_{0<u<x}\beta(u)]$. This is unbounded if β is unbounded near 0.

A second result in this context deals with the extremal zeros. Here we need the extra conditions that (2.6) also holds at $x = 0$, that the limit functions a and b are continuous at $x = 0$ and that $a + b$ is a positive function on $[0, +\infty)$.

Theorem 2.6. *Let for each $N \in \mathbb{N}$ two sequences $\{a_{n,N}\}_{n\geqslant 1}$ and $\{b_{n,N}\}_{n\geqslant 1}$ of positive recurrence coefficients be given and let $\{P_{n,N}\}$ be the polynomials satisfying (2.5). Suppose there exist two continuous functions a:$[0, +\infty) \to [0, +\infty)$ and b:$[0, +\infty) \to [0, +\infty)$, with $a + b > 0$, for which*

$$\lim_{n/N\to x} a_{n,N} = a(x), \qquad \lim_{n/N\to x} b_{n,N} = b(x), \qquad x \geqslant 0, \tag{2.8}$$

and define the functions α and β by transformation (1.11) for $x \geqslant 0$. Furthermore, let $y_1(n, N)$ and $y_n(n, N)$ denote the smallest and largest zero of $P_{n,N}$, respectively. We then have

$$\lim_{n/N\to x} y_1(n, N) = \min_{0\leqslant u\leqslant x} \alpha(u), \tag{2.9}$$

$$\lim_{n/N\to x} y_n(n, N) = \max_{0\leqslant u\leqslant x} \beta(u), \tag{2.10}$$

for every $x > 0$.

In section 2.2 we will discuss a general theorem on ratio asymptotics for monic Laurent orthogonal polynomials with varying recurrence coefficients. This will help us to prove theorem 2.3 in section 2.3. Finally, we also prove theorem 2.6.

2.2. Ratio asymptotics

We look for the ratio asymptotics of the monic Laurent orthogonal polynomials $P_{n,N}$, defined by the recurrence relation (2.5) with varying recurrence coefficients. Here we use similar techniques as in [22] for the case of orthogonal polynomials.

Theorem 2.7. *Suppose we have for each $N \in \mathbb{N}$ two sequences $\{a_{n,N}\}_{n\geqslant 1}$ and $\{b_{n,N}\}_{n\geqslant 1}$ of positive recurrence coefficients and let $\{P_{n,N}\}$ be the doubly indexed sequence of polynomials defined by recurrence (2.5). Assume for some fixed $x > 0$ that*

$$\lim_{n/N\to x} a_{n,N} = a(x) \geqslant 0, \qquad \lim_{n/N\to x} b_{n,N} = b(x) \geqslant 0, \tag{2.11}$$

and define $\alpha(x)$ and $\beta(x)$ by transformation (1.11). Furthermore, assume that for some $x^\star > x$ there exist $r_1, r_2 \in \mathbb{R}$ such that $r_1 \leqslant \alpha(x) \leqslant \beta(x) \leqslant r_2$ and that all the zeros of $P_{n,N}$ belong to $[r_1, r_2]$ whenever $n \leqslant x^\star N$. Then

$$\lim_{n/N\to x} \frac{P_{n-1,N}(z)}{P_{n,N}(z)} = \frac{2}{z - a(x) + \sqrt{(z - a(x))^2 - 4zb(x)}} \tag{2.12}$$

uniformly on compact subsets of $\mathbb{C}\backslash[r_1, r_2]$. Here the square root is such that $\sqrt{(z - a(x))^2 - 4zb(x)} = \sqrt{(z - \alpha(x))(z - \beta(x))}$ is an analytic function of z in $\mathbb{C}\backslash[\alpha(x), \beta(x)]$, which is positive for $z > \beta(x)$.

Remark 2.8. This theorem generalizes [31, theorem 4.1] in the case that the $a_{n,N}$ and $b_{n,N}$ do not depend on N.

To prove this theorem we need the following lemma, which will be used in the proof of theorem 2.3 as well. It can be found in [22, lemma 2.2], but we include a short proof for completeness.

Lemma 2.9. *Suppose that the zeros of the monic polynomials p_{n-1} and p_n, with degree $n-1$ and n, respectively, are simple and real, interlace and lie in $[r_1, r_2]$. Then*

$$(a)\qquad \left|\frac{p_{n-1}(z)}{p_n(z)}\right| \leqslant \frac{1}{\operatorname{dist}(z,[r_1,r_2])}, \qquad \forall z \in \mathbb{C}\backslash[r_1,r_2], \tag{2.13}$$

$$(b)\qquad \left|\frac{p_{n-1}(z)}{p_n(z)}\right| \geqslant \frac{1}{2|z|}, \qquad \textit{if}\quad |z| > \max(|r_1|,|r_2|). \tag{2.14}$$

Proof. Denote the real zeros of p_n by $y_1,\ldots,y_n$. Since p_{n-1} and p_n are monic and their zeros interlace, there exist $c_j > 0$, $\sum_{j=1}^n c_j = 1$, so that

$$\frac{p_{n-1}(z)}{p_n(z)} = \sum_{j=1}^n \frac{c_j}{z-y_j}.$$

Then note that, because $y_j \in [r_1, r_2]$, for all $z \in \mathbb{C}\backslash[r_1,r_2]$ we have $|z-y_j| \geqslant \operatorname{dist}(z,[r_1,r_2])$, $1 \leqslant j \leqslant n$. This immediately proves (2.13).

If $|z| > \max(|r_1|,|r_2|)$, then $|y_j/z| < 1$ and therefore $\Re\left(\frac{1}{1-y_j/z}\right) > \frac{1}{2}$, $1 \leqslant j \leqslant n$. Hence

$$\frac{1}{|z|}\left|\sum_{j=1}^n \frac{c_j}{1-y_j/z}\right| \geqslant \frac{1}{|z|}\Re\left(\sum_{j=1}^n \frac{c_j}{1-y_j/z}\right) > \frac{1}{2|z|}\sum_{j=1}^n c_j = \frac{1}{2|z|},$$

which proves (2.14). □

We are now ready to prove theorem 2.7.

Proof of theorem 2.7. For the zeros of the (Laurent orthogonal) polynomials $P_{n,N}$ defined by recurrence (2.5) it is known that they are positive and simple and that, for fixed N, they satisfy the interlacing property, see, e.g., [9, 31]. As a corollary, every member of

$$\left\{\frac{P_{n-1,N}(z)}{P_{n,N}(z)} : n, N \in \mathbb{N}, n \leqslant x^{\star}N\right\}, \tag{2.15}$$

satisfies the estimate (2.13), independent of n and N. So, the family of functions (2.15) is uniformly bounded on compact subsets of $\overline{\mathbb{C}}\backslash[r_1,r_2]$ and is hence a normal family on $\overline{\mathbb{C}}\backslash[r_1,r_2]$. For each sequence $\{(n_j,N_j)\}_{j\geqslant 1}$, with $n_j, N_j \to \infty$, $n_j/N_j \to x$ as $j\to\infty$, we have that, if j is sufficiently large, the functions

$$f_j(z) := \frac{P_{n_j-1,N}(z)}{P_{n_j,N}(z)} \tag{2.16}$$

belong to this normal family (2.15). The sequence $\{f_j\}_{j\geqslant 1}$ then has a subsequence that converges uniformly on compact subsets of $\overline{\mathbb{C}}\backslash[r_1,r_2]$. If we can prove that the corresponding limit of any such subsequence is

$$\phi(z) := \frac{2}{z-a(x)+\sqrt{(z-a(x))^2-4zb(x)}}, \tag{2.17}$$

then, by a standard compactness argument, the full sequence $\{f_j\}_{j\geqslant 1}$ converges uniformly on compact subsets of $\overline{\mathbb{C}}\backslash[r_1,r_2]$ to ϕ. This then proves the theorem.

We now prove that for each sequence $\{(n_i,N_i)\}_{i\geqslant 1}$ such that $n_i, N_i \to \infty$, $n_i/N_i \to x$ and the functions $\{f_i\}_{i\geqslant 1}$ converge uniformly on compact subsets of $\overline{\mathbb{C}}\backslash[r_1,r_2]$ as $i\to\infty$, we have

$$f(z) := \lim_{i\to\infty} f_i(z) = \phi(z) + \mathcal{O}(z^{-k}), \qquad \text{as } z\to\infty, \tag{2.18}$$

for each $k \in \mathbb{N}$. The uniqueness of the Laurent expansion around infinity then implies that $f(z) = \phi(z)$. We show this by induction on k. For the case $k = 1$ we just observe that $\phi(z) = z^{-1} + \mathcal{O}(z^{-2})$ and $f_j(z) = \mathcal{O}(z^{-1})$, for every j. Next, suppose that this claim is true for some $k = \ell \geqslant 1$ and consider a sequence $\{(n_i, N_i)\}_{i\geqslant 1}$ such that $n_i, N_i \to \infty$, $n_i/N_i \to x$ and the functions $\{f_i\}_{i\geqslant 1}$ converge uniformly on compact subsets of $\overline{\mathbb{C}}\backslash[r_1, r_2]$ to some function f as $i \to \infty$. If we define

$$g_i(z) := \frac{P_{n_i-2,N}(z)}{P_{n_i-1,N}(z)}, \qquad z \in \mathbb{C}\backslash[r_1, r_2],$$

then we obtain from recurrence (2.5) that

$$f_i(z)^{-1} = z - a_{n_i,N_i} - b_{n_i-1,N_i} z g_i(z). \tag{2.19}$$

Note that since $x < x^\star$ we may assume without loss of generality that $n_i < x^\star N_i$, and so $n_i - 1 < x^\star N_i$, for every i. This means that $\{g_i\}_{i\geqslant 1}$ is a subset of the normal family (2.15) and therefore, it has a subsequence that converges uniformly on compact subsets of $\overline{\mathbb{C}}\backslash[r_1, r_2]$ to some function g for which $g(z) = \phi(z) + \mathcal{O}(z^{-\ell})$ as $z \to \infty$ by the induction hypothesis. Passing to such a subsequence and taking $i \to \infty$ in (2.19) we get

$$f(z)^{-1} = z - a(x) - b(x)zg(z) = z - a(x) - b(x)z\phi(z) + \mathcal{O}(z^{-\ell+1}), \qquad \text{as } z \to \infty.$$

It is easy to see that

$$\begin{aligned} z - a(x) - b(x)z\phi(z) &= \frac{(z-a(x))\left(\frac{z-a(x)}{2} + \sqrt{\left(\frac{z-a(x)}{2}\right)^2 - zb(x)}\right) - b(x)z}{\frac{z-a(x)}{2} + \sqrt{\left(\frac{z-a(x)}{2}\right)^2 - zb(x)}} \\ &= \frac{\left(\frac{z-a(x)}{2}\right)^2 + 2\left(\frac{z-a(x)}{2}\right)\sqrt{\left(\frac{z-a(x)}{2}\right)^2 - zb(x)} + \left(\frac{z-a(x)}{2}\right)^2 - b(x)z}{\frac{z-a(x)}{2} + \sqrt{\left(\frac{z-a(x)}{2}\right)^2 - zb(x)}} = \phi(z)^{-1}, \end{aligned} \tag{2.20}$$

so that $f(z)^{-1} = \phi(z)^{-1} + \mathcal{O}(z^{-\ell+1})$. Now recall that $\phi(z) = z^{-1} + \mathcal{O}(z^{-2})$, so that we finally obtain

$$f(z) = \frac{\phi(z)}{1 + \phi(z)\mathcal{O}(z^{-\ell+1})} = \frac{\phi(z)}{1 + \mathcal{O}(z^{-\ell})} = \phi(z)(1 + \mathcal{O}(z^{-\ell})) = \phi(z) + \mathcal{O}(z^{-\ell-1}),$$

as $z \to \infty$, which completes the proof by induction. □

2.3. *Proofs of theorems 2.3 and 2.6*

First of all we give a lemma dealing with the logarithmic potential of the probability measure $\upsilon_{[\alpha,\beta]}$. This will help us to prove theorem 2.3.

Lemma 2.10. *Let $\upsilon_{[\alpha,\beta]}$ be the measure defined by (2.4) if $0 < \alpha < \beta$, by $\frac{1}{2}\omega_{[0,\beta]} + \frac{1}{2}\delta_0$ if $0 = \alpha < \beta$ and by δ_α if $0 \leqslant \alpha = \beta$. Here $\omega_{[0,\beta]}$ is the arcsine measure on $[0, \beta]$. For its logarithmic potential we then have*

$$U^{\upsilon_{[\alpha,\beta]}}(z) = -\log\left|\frac{z - \sqrt{\alpha\beta} + \sqrt{(z-\alpha)(z-\beta)}}{2}\right|, \qquad z \in \mathbb{C}\backslash[\alpha, \beta], \tag{2.21}$$

where the square root is such that $\sqrt{(z-\alpha)(z-\beta)}$ is an analytic function of z in $\mathbb{C}\backslash[\alpha, \beta]$, which is positive for $z > \beta$. In the case $0 < \alpha < \beta$ we also have the expression

$$U^{\upsilon_{[\alpha,\beta]}}(z) = -\log\frac{\sqrt{\beta} - \sqrt{\alpha}}{2} + \frac{1}{2}\log\frac{1}{|z|} - \frac{1}{2}g_{[\alpha,\beta]}(z, \infty) - \frac{1}{2}g_{[\alpha,\beta]}(z, 0), \qquad z \in \mathbb{C}, \tag{2.22}$$

where $g_{[\alpha,\beta]}(z,\infty)$ *and* $g_{[\alpha,\beta]}(z,0)$ *are the Green functions for the complement of* $[\alpha,\beta]$ *with pole at* ∞ *and 0, respectively.*

Proof. The case $0 \leqslant \alpha = \beta$ is trivial since $U^{\delta_\alpha}(z) = -\log|z-\alpha|$. We now study the case $0 < \alpha < \beta$ for which we noted in remark 2.2 that $\upsilon_{[\alpha,\beta]} = \frac{1}{2}\omega_{[\alpha,\beta]} + \frac{1}{2}\hat{\delta}_{0,[\alpha,\beta]}$, with $\omega_{[\alpha,\beta]}$ the arcsine measure on $[\alpha,\beta]$ and $\hat{\delta}_{0,[\alpha,\beta]}$ the balayage of the Dirac measure δ_0 on $[\alpha,\beta]$. By [32, (4.2), p 108] and [32, chapter I, example 3.5] we know that

$$U^{\omega_{[\alpha,\beta]}}(z) = -\log\frac{\beta-\alpha}{4} - g_{[\alpha,\beta]}(z,\infty), \tag{2.23}$$

with

$$g_{[\alpha,\beta]}(z,\infty) = \begin{cases} \log\left|\dfrac{2z-(\alpha+\beta)+2\sqrt{(z-\alpha)(z-\beta)}}{\beta-\alpha}\right|, & z\in\mathbb{C}\backslash[\alpha,\beta],\\ 0, & z\in[\alpha,\beta], \end{cases} \tag{2.24}$$

where the square root is such that $\sqrt{(z-\alpha)(z-\beta)}$ is an analytic function of z in $\mathbb{C}\backslash[\alpha,\beta]$, which is positive for $z>\beta$ and hence negative for $z<\alpha$. Furthermore, from [32, (4.32), p 119] we obtain

$$U^{\hat{\delta}_{0,[\alpha,\beta]}}(z) = g_{[\alpha,\beta]}(0,\infty) + \log\frac{1}{|z|} - g_{[\alpha,\beta]}(z,0), \tag{2.25}$$

where, by evaluating (2.24) at $z=0$,

$$g_{[\alpha,\beta]}(0,\infty) = \log\left|\frac{-(\alpha+\beta)-2\sqrt{\alpha\beta}}{\beta-\alpha}\right| = \log\frac{\sqrt{\beta}+\sqrt{\alpha}}{\sqrt{\beta}-\sqrt{\alpha}} \tag{2.26}$$

since $0<\alpha$. Combining (2.23), (2.25) and (2.26) we then find (2.22).

Using [32, (4.4), p 109] and (2.24) we get

$$g_{[\alpha,\beta]}(z,0) = g_{[\frac{1}{\beta},\frac{1}{\alpha}]}\left(\frac{1}{z},\infty\right) = \log\left|\frac{(\alpha+\beta)z-2\alpha\beta+2\sqrt{\alpha\beta}\sqrt{(z-\alpha)(z-\beta)}}{z(\beta-\alpha)}\right|, \tag{2.27}$$

for $z\in\mathbb{C}\backslash[\alpha,\beta]$, where the square root is such that $\sqrt{(z-\alpha)(z-\beta)}$ is an analytic function of z in $\mathbb{C}\backslash[\alpha,\beta]$, which is positive for $z>\beta$. Combining (2.22), (2.24) and (2.27), for the case $0<\alpha<\beta$ we then establish

$$\begin{aligned} U^{\upsilon_{[\alpha,\beta]}}(z) = &-\frac{1}{2}\log\left|\frac{z-\frac{\alpha+\beta}{2}+\sqrt{(z-\alpha)(z-\beta)}}{\sqrt{\beta}+\sqrt{\alpha}}\right|\\ &-\frac{1}{2}\log\left|\frac{\frac{\alpha+\beta}{2}z-\alpha\beta+\sqrt{\alpha\beta}\sqrt{(z-\alpha)(z-\beta)}}{\sqrt{\beta}+\sqrt{\alpha}}\right|, \qquad z\in\mathbb{C}\backslash[\alpha,\beta]. \end{aligned} \tag{2.28}$$

Now define $a=\sqrt{\alpha\beta}$, $b=(\sqrt{\beta}-\sqrt{\alpha})^2/4$, so that $4(a+b)=(\sqrt{\beta}+\sqrt{\alpha})^2$ and $a+2b=\frac{\alpha+\beta}{2}$. Next, we also introduce the function

$$\phi(z) := \frac{2}{z-\sqrt{\alpha\beta}+\sqrt{(z-\alpha)(z-\beta)}}, \qquad z\in\mathbb{C}\backslash[\alpha,\beta].$$

Replacing $\sqrt{(z-\alpha)(z-\beta)}$ by $2\phi(z)^{-1}-z+a$ in (2.28) we then get

$$\begin{aligned} U^{\upsilon_{[\alpha,\beta]}}(z) &= -\frac{1}{2}\log\left|\frac{(\phi(z)^{-1}-b)(bz+a\phi(z)^{-1})}{a+b}\right|\\ &= -\frac{1}{2}\log\left|\frac{bz\phi(z)+a-b\phi(z)(bz\phi(z)+a)}{\phi(z)^2(a+b)}\right|, \qquad z\in\mathbb{C}\backslash[\alpha,\beta]. \end{aligned} \tag{2.29}$$

For the function ϕ we proved in (2.20) that $bz\phi(z)+a = z-\phi(z)^{-1}$, so we finally obtain $U^{\upsilon_{[\alpha,\beta]}}(z) = \log|\phi(z)|$, $z\in\mathbb{C}\backslash[\alpha,\beta]$, which is equal to (2.21).

For the case $0=\alpha<\beta$ we mention that expressions (2.26) and (2.27) are equal to 0 if $\alpha=0$ and that $U^{\delta_0}(z) = -\log|z|$. So, the above computations still hold. □

We now prove theorems 2.3 and 2.6.

Proof of theorem 2.3. Let $x > 0$. First we prove the theorem under the additional assumption that, for a number $x^\star > x$,

$$0 < r := 6 \sup\{a_{n,N}, b_{n,N} : n \leqslant x^\star N\} < \infty. \tag{2.30}$$

Clearly, the convergence (2.6) and the fact that the functions a, b are continuous on $(0, \infty)$ imply that the recurrence coefficients are uniformly bounded if n/N is restricted to compact subsets of $(0, \infty)$. So, this condition deals with the behaviour of the recurrence coefficients for small n/N.

Define for each $n, N \in \mathbb{N}$ the function $Q_{n,N}(z) := P_{n,N}(z)/z^n$. We know that the zeros of $P_{n,N}$ are simple and positive, and so are the n zeros of $Q_{n,N}$. It is easily verified that these functions satisfy the recurrence

$$z(Q_{n-1,N}(z) - Q_{n,N}(z)) = a_{n,N} Q_{n-1,N}(z) + b_{n-1,N} Q_{n-2,N}(z), \qquad n \geqslant 1,$$

with the initial conditions $Q_{0,N} \equiv 1$, $Q_{-1,N} \equiv 0$. This implies that the zeros of $P_{n,N}$ are also the generalized eigenvalues for the pair of matrices $(F_{n,N}, G_{n,N})$, with

$$F_{n,N} = \begin{pmatrix} a_{1,N} & 0 & \cdots & \cdots & 0 \\ b_{1,N} & a_{2,N} & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & b_{n-1,N} & a_{n,N} \end{pmatrix}, \qquad G_{n,N} = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & 1 & -1 \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}. \tag{2.31}$$

If λ is such an eigenvalue, then $\det(\lambda G_{n,N} - F_{n,N}) = 0$. Now define for some $\kappa > 0$ the diagonal matrix $D_{n,\kappa} := \operatorname{diag}(\kappa, \kappa^2, \ldots, \kappa^n)$, then also $\det(D_{n,\kappa}(\lambda G_{n,N} - F_{n,N}) D_{n,\kappa}^{-1}) = 0$. This means that the matrix $D_{n,\kappa}(\lambda G_{n,N} - F_{n,N}) D_{n,\kappa}^{-1}$ cannot be strictly diagonally dominant. Since we know that λ, κ and $b_{i-1,N}$ are positive, we see that at least one of the conditions

$$|\lambda - a_{i,N}| \leqslant \frac{\lambda}{\kappa} + \kappa b_{i-1,N}, \qquad 1 \leqslant i \leqslant n,$$

must hold, where we take $b_{0,N} := b_{1,N}$. Assuming $n \leqslant x^\star N$ and $\kappa > 1$, by (2.30) we then get that λ has the upper bound

$$\lambda \leqslant \max_{1 \leqslant i \leqslant n} \frac{a_{i,N} + \kappa b_{i-1,N}}{1 - \frac{1}{\kappa}} \leqslant \frac{r}{6} \frac{\kappa(1+\kappa)}{\kappa - 1}.$$

By taking $\kappa = 3$, we finally get that, under the additional assumption (2.30), the zeros of the Laurent orthogonal polynomials $P_{n,N}$, $n \leqslant x^\star N$, are all in the interval $[0, r]$.

Observe that

$$\begin{aligned} 0 \leqslant \alpha(ux) \leqslant \beta(ux) &= \left(\sqrt{a(ux) + b(ux)} + \sqrt{b(ux)}\right)^2 \\ &\leqslant r \left(\tfrac{1}{\sqrt{3}} + \tfrac{1}{\sqrt{6}}\right)^2 \\ &< r, \end{aligned} \tag{2.32}$$

for each $0 < u \leqslant 1$. Hence from (2.6) and theorem 2.7 we establish

$$\lim_{n/N \to x} \frac{P_{\lceil un \rceil, N}(z)}{P_{\lceil un \rceil - 1, N}(z)} = \frac{z - \sqrt{\alpha(ux)\beta(ux)} + \sqrt{(z - \alpha(ux))(z - \beta(ux))}}{2}, \tag{2.33}$$

for every $z \in \mathbb{C}\backslash[0, r]$, where $0 < u \leqslant 1$ and $\lceil un \rceil$ denotes the smallest integer greater than or equal to un. Note that

$$P_{n,N}(z) = \prod_{k=1}^{n} \frac{P_{k,N}(z)}{P_{k-1,N}(z)}, \qquad n \in \mathbb{N},$$

so that

$$\frac{1}{n} \log |P_{n,N}(z)| = \frac{1}{n} \sum_{k=1}^{n} \log \left| \frac{P_{k,N}(z)}{P_{k-1,N}(z)} \right| = \int_0^1 \log \left| \frac{P_{\lceil un \rceil,N}(z)}{P_{\lceil un \rceil-1,N}(z)} \right| \mathrm{d}u. \tag{2.34}$$

We already mentioned that the zeros of the polynomials $P_{n,N}$ are positive and simple and that, for fixed N, they satisfy the interlacing property. If $n \leqslant x^\star N$, then from lemma 2.9 we get

$$\mathrm{dist}(z, [0, r]) \leqslant \left| \frac{P_{\lceil un \rceil,N}(z)}{P_{\lceil un \rceil-1,N}(z)} \right| \leqslant 2|z|, \qquad |z| > r,$$

and so

$$\left| \log \left| \frac{P_{\lceil un \rceil,N}(z)}{P_{\lceil un \rceil-1,N}(z)} \right| \right| \leqslant \max(|\log(\mathrm{dist}(z, [0, r]))|, |\log(2|z|)|), \tag{2.35}$$

for $|z| > r$. By (2.33), applying Lebesgue's dominated convergence theorem on (2.34) then gives, for $|z| > r$,

$$\begin{aligned} \lim_{n/N \to x} \frac{1}{n} \log |P_{n,N}(z)| &= \int_0^1 \log \left| \frac{z - \sqrt{\alpha(ux)\beta(ux)} + \sqrt{(z - \alpha(ux))(z - \beta(ux))}}{2} \right| \mathrm{d}u \\ &= \frac{1}{x} \int_0^x \log \left| \frac{z - \sqrt{\alpha(u)\beta(u)} + \sqrt{(z - \alpha(u))(z - \beta(u))}}{2} \right| \mathrm{d}u. \end{aligned}$$

Because of (2.32) and lemma 2.10 we can conclude that, for $|z| > r$,

$$\lim_{n/N \to x} U^{\nu(P_{n,N})}(z) = \lim_{n/N \to x} \frac{1}{n} \log \frac{1}{|P_{n,N}(z)|} = \frac{1}{x} \int_0^x U^{\upsilon_{[\alpha(u),\beta(u)]}}(z)\, \mathrm{d}u = U^{\nu_x}(z), \tag{2.36}$$

where the measure ν_x acts on an arbitrary Borel set E like

$$\nu_x(E) = \frac{1}{x} \int_0^x \upsilon_{[\alpha(u),\beta(u)]}(E)\, \mathrm{d}u.$$

Since for $n \leqslant x^\star N$ the zeros of $P_{n,N}$ are in $[0, r]$, any sequence $\{\nu(P_{n_j,N_j})\}_{j\geqslant 1}$ for which $n_j \to \infty$, $N_j \to \infty$ and $n_j/N_j \to x$, as $j \to \infty$, contains a converging subsequence (in the sense of weak-$\star$ convergence). Suppose μ_x is the limit of such a converging subsequence. Then $\mathrm{supp}(\mu_x) \subset [0, r]$ and so, from (2.36) we get

$$U^{\mu_x}(z) = U^{\nu_x}(z), \qquad |z| > r, \tag{2.37}$$

since $-\log|z - y|$ is a continuous function on $[0, r]$ if $|z| > r$. Note that also $\mathrm{supp}(\nu_x) = \left[\inf_{0<u<x} \alpha(u), \sup_{0<u<x} \beta(u)\right] \subset [0, r]$. Both sides of (2.37) are thus harmonic on $\mathbb{C}\backslash[0, r]$, implying that equality (2.37) holds for $z \in \mathbb{C}\backslash[0, r]$. By the unicity theorem of potentials, see, e.g., [32, chapter 2, corollary 2.2], we then obtain $\mu_x = \nu_x$. So, by [3, theorem 2.3] this proves (2.7) under the additional assumption (2.30).

We now prove the theorem in the general case. Here we need the ith associated polynomials, with $i \in \mathbb{N}$, defined by the recurrence relation

$$P_{n,N}^{(i)}(z) = \left(z - a_{n,N}^{(i)}\right) P_{n-1,N}^{(i)}(z) - b_{n-1,N}^{(i)} z P_{n-2,N}^{(i)}(z), \qquad n \geqslant 1, \tag{2.38}$$

with $a^{(i)}_{n,N} := a_{n+i,N}$, $b^{(i)}_{n,N} := b_{n+i,N}$, and the initial conditions $P^{(i)}_{0,N} \equiv 1$, $P^{(i)}_{-1,N} \equiv 0$. If we multiply (2.5) by $P^{(i)}_{n-i-1,N}(z)$ and (2.38), after substituting $n \to n-i$, by $P_{n-1,N}(z)$ and subtract the two obtained equations, it is easy to see by induction on n that

$$P_{n-1,N}(z)P^{(i)}_{n-i,N}(z) - P^{(i)}_{n-i-1,N}(z)P_{n,N}(z) = z^{n-i}P_{i-1,N}(z)\prod_{k=i}^{n-1} b_{k,N}, \qquad n \geqslant i.$$

So, it turns out that at each (positive) zero of $P_{n,N}/P_{i-1,N}$ the polynomials $P_{n-1,N}$ and $P^{(i)}_{n-i,N}P_{i-1,N}$ have the same sign. Since the zeros of $P_{n,N}$ and $P_{n-1,N}$ interlace, between each two successive zeros of $P_{n,N}/P_{i-1,N}$ there is just one zero of $P^{(i)}_{n-i,N}P_{i-1,N}/P_{n,N}$. Then there exist $n-i-1$ zeros of $P_{n,N}$ which separate those of $P^{(i)}_{n-i,N}$.

Let $\delta > 0$ be fixed. From (2.6) we then obtain that

$$\lim_{n/N\to x} a^{(\lfloor\delta N\rfloor)}_{n,N} = a(x+\delta), \qquad \lim_{n/N\to x} b^{(\lfloor\delta N\rfloor)}_{n,N} = b(x+\delta), \qquad x > 0.$$

Furthermore, the functions a and b are continuous. So, the set of recurrence coefficients $a^{(\lfloor\delta N\rfloor)}_{n,N}$, $b^{(\lfloor\delta N\rfloor)}_{n,N}$, with $n \leqslant x^\star N$, is uniformly bounded for every $x^\star > 0$. Hence, from the first part of this proof we obtain, for $x > \delta$,

$$\begin{aligned}\lim_{n/N\to x} \nu\big(P^{(\lfloor\delta N\rfloor)}_{n-\lfloor\delta N\rfloor,N}\big) &= \frac{1}{x-\delta}\int_0^{x-\delta} \upsilon_{[\alpha(u+\delta),\beta(u+\delta)]}\,\mathrm{d}u \\ &= \frac{1}{x-\delta}\int_\delta^{x} \upsilon_{[\alpha(u),\beta(u)]}\,\mathrm{d}u.\end{aligned} \tag{2.39}$$

Note that for N large enough we have $\lfloor\delta N\rfloor \geqslant 1$. Then, for each n, N, with $N > 1/\delta$, there are $n - \lfloor\delta N\rfloor - 1$ zeros of $P_{n,N}$ which separate the zeros of $P^{(i)}_{n-\lfloor\delta N\rfloor,N}$. These zeros then also have the asymptotic distribution (2.39) as $n/N \to x$ and, as a consequence, they give the contribution

$$\frac{1-\frac{\delta}{x}}{x-\delta}\int_\delta^{x} \upsilon_{[\alpha(u),\beta(u)]}\,\mathrm{d}u = \frac{1}{x}\int_\delta^{x} \upsilon_{[\alpha(u),\beta(u)]}\,\mathrm{d}u \tag{2.40}$$

to the asymptotic zero distribution of the $P_{n,N}$ as $n/N \to x$.

The contribution of the remaining $\lfloor\delta N\rfloor + 1 \approx (\delta/x)n$ zeros of $P_{n,N}$ to the asymptotic zero distribution of the $P_{n,N}$ is negligible as $\delta \to 0$. So, letting δ tend to 0 in (2.40) then completes the proof. □

Proof of theorem 2.6. Let $x > 0$. Since the functions α and β are continuous on $[0, x]$, from theorem 2.3 and remark 2.5 it is clear that the zeros of the polynomials $P_{n,N}$ are dense in the interval $[\min_{0\leqslant u\leqslant x}\alpha(u), \max_{0\leqslant u\leqslant x}\beta(u)]$ whenever $N \to \infty$ and $n/N \to x$. This already implies

$$\limsup_{n/N\to x} y_1(n,N) \leqslant \min_{0\leqslant u\leqslant x}\alpha(u), \qquad \liminf_{n/N\to x} y_n(n,N) \geqslant \max_{0\leqslant u\leqslant x}\beta(u).$$

Let $\kappa^{(i)}_N > 0$, $i \geqslant 1$ and $D_{n,\vec{\kappa}_N} := \mathrm{diag}\big(\kappa^{(1)}_N, \kappa^{(1)}_N\kappa^{(2)}_N, \ldots, \prod_{i=1}^n \kappa^{(i)}_N\big)$ for each $n, N \in \mathbb{N}$. As shown in the proof of theorem 2.3, the zeros of $P_{n,N}$ are also the generalized eigenvalues for the pair of matrices $(F_{n,N}, G_{n,N})$, defined as in (2.31). So, if λ is such an eigenvalue, then $\det\big(D_{n,\vec{\kappa}_N}(\lambda G_{n,N} - F_{n,N})D^{-1}_{n,\vec{\kappa}_N}\big) = 0$. As an easy consequence the matrix $D_{n,\vec{\kappa}_N}(\lambda G_{n,N} - F_{n,N})D^{-1}_{n,\vec{\kappa}_N}$ cannot be strictly diagonally dominant, giving

$$\exists\, i \in \{1,\ldots,n\} \quad \text{such that} \quad |\lambda - a_{i,N}| \leqslant \frac{\lambda}{\kappa^{(i)}_N} + \kappa^{(i-1)}_N b_{i-1,N}, \tag{2.41}$$

where we take $b_{0,N} := b_{1,N}$, $\kappa^{(0)}_N := \kappa^{(1)}_N$.

First of all, with the choice

$$1 < \kappa_N^{(i)} := 1 + \sqrt{\frac{a_{i,N} + b_{i,N}}{b_{i,N}}}, \qquad i \geqslant 1,$$

where $\kappa_N^{(0)} := \kappa_N^{(1)}$ implies $a_{0,N} := a_{1,N}$, from (2.41) we obtain

$$\begin{aligned} y_n(n, N) &\leqslant \max_{1\leqslant i\leqslant n} \frac{\kappa_N^{(i)}}{\kappa_N^{(i)} - 1}\left(a_{i,N} + \kappa_N^{(i-1)} b_{i-1,N}\right) \\ &= \max_{1\leqslant i\leqslant n} \left(\sqrt{a_{i,N} + b_{i,N}} + \sqrt{b_{i,N}}\right) \left(\frac{a_{i,N} + b_{i-1,N}}{\sqrt{a_{i,N} + b_{i,N}}} + \sqrt{b_{i-1,N}}\sqrt{\frac{a_{i-1,N} + b_{i-1,N}}{a_{i,N} + b_{i,N}}}\right). \end{aligned}$$

From the assumptions of the theorem (limits (2.8) and the function $a + b$ has no zeros), we then immediately get $\limsup_{n/N\to x} y_n(n, N) \leqslant \max_{0\leqslant u\leqslant x} \beta(u)$, which completes the proof of (2.10).

Secondly, choose

$$0 < \kappa_N^{(i)} := -1 + \sqrt{\frac{a_{i,N} + b_{i,N}}{b_{i,N}}}, \qquad i \geqslant 1,$$

where $\kappa_N^{(0)} := \kappa_N^{(1)}$ again implies $a_{0,N} := a_{1,N}$. From (2.41) we then establish

$$\begin{aligned} y_1(n, N) &\geqslant \min_{1\leqslant i\leqslant n} \frac{\kappa_N^{(i)}}{\kappa_N^{(i)} + 1}\left(a_{i,N} - \kappa_N^{(i-1)} b_{i-1,N}\right) \\ &= \min_{1\leqslant i\leqslant n} \left(\sqrt{a_{i,N} + b_{i,N}} - \sqrt{b_{i,N}}\right) \left(\frac{a_{i,N} + b_{i-1,N}}{\sqrt{a_{i,N} + b_{i,N}}} - \sqrt{b_{i-1,N}}\sqrt{\frac{a_{i-1,N} + b_{i-1,N}}{a_{i,N} + b_{i,N}}}\right), \end{aligned}$$

which gives $\liminf_{n/N\to x} y_1(n, N) \geqslant \min_{0\leqslant u\leqslant x} \alpha(u)$ by the assumptions of the theorem. This finally proves (2.9). □

3. Direct problem for the continuum limit of the RTL

3.1. Definition of the spectral data

In the direct problem we start from operator data $a, b \in C[0, 1]$ satisfying

$$\begin{aligned} &a(x) > 0, \qquad 0 \leqslant x \leqslant 1, \\ &b(x) > 0, \qquad 0 < x < 1, \qquad b(0) = b(1) = 0. \end{aligned}$$

For each $N \in \mathbb{N}$ we define the sets of discrete operator data

$$\begin{aligned} a_{k,N} &= a(k/N) > 0, \qquad 1 \leqslant k \leqslant N, \\ b_{k,N} &= b(k/N) > 0, \qquad 1 \leqslant k \leqslant N - 1, \end{aligned}$$

implying the limits

$$\lim_{n/N\to x} a_{n,N} = a(x), \qquad \lim_{n/N\to x} b_{n,N} = b(x), \qquad 0 \leqslant x \leqslant 1.$$

Using the spectral transform for the discrete finite RTL, described in section 1.1, we obtain sets of discrete spectral data $0 < \lambda_{1,N} < \cdots < \lambda_{N,N}$ and $w_{j,N} > 0$, $\sum_{j=1}^{N} w_{j,N} = 1$, for each $N \in \mathbb{N}$. From these, we define the spectral data for the continuum limit as follows. First, we consider the measure $\sigma \in \mathcal{M}_1(\mathbb{R}^+)$, where $\mathcal{M}_1(\mathbb{R}^+)$ is the set of Borel probability measures on $\mathbb{R}^+$, satisfying

$$\sigma := \lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} \delta_{\lambda_{j,N}} \tag{3.1}$$

in the sense of weak-⋆ convergence. This limit indeed exists and in section 2 we established an explicit formula in terms of the operator data. Secondly, we assume that there exists a function R, defined on $\operatorname{supp}(\sigma)$, satisfying

$$\lim_{N\to\infty} \max_{1\leqslant j\leqslant N} \left| \frac{1}{N} \log w_{j,N} - R(\lambda_{j,N}) \right| = 0. \tag{3.2}$$

Since it is not so clear that this limit always exists, this includes an extra condition on the operator data.

3.2. Spectral data in terms of the operator data

Recall that, for each $N \in \mathbb{N}$, the $\lambda_{j,N}$ are the zeros of the polynomial $P_{N,N}$, defined by the recurrence relation

$$P_{n,N}(z) = (z - a_{n,N})P_{n-1,N}(z) - b_{n-1,N}zP_{n-2,N}(z), \qquad 1 \leqslant n \leqslant N, \tag{3.3}$$

with initial conditions $P_{0,N} \equiv 1$, $P_{-1,N} \equiv 0$. The next result for the measure σ then follows from theorem 2.3.

Theorem 3.1. *Limit (3.1) exists and has the expression*

$$\sigma = \int_0^1 \upsilon_{[\alpha(x),\beta(x)]}\, \mathrm{d}x, \tag{3.4}$$

where the functions α, β are constructed from a and b by transformation (1.11) and, for $0 < \alpha < \beta$,

$$\frac{\mathrm{d}\upsilon_{[\alpha,\beta]}(\lambda)}{\mathrm{d}\lambda} = \begin{cases} \dfrac{1}{2\pi} \dfrac{\lambda + \sqrt{\alpha\beta}}{\lambda\sqrt{(\beta-\lambda)(\lambda-\alpha)}}, & \lambda \in [\alpha, \beta], \\ 0, & \textit{elsewhere}, \end{cases} \tag{3.5}$$

which is the minimizer for the logarithmic energy in the external field $\frac{1}{2}\log y$ among the probability measures on $[\alpha, \beta]$ (see remark 2.2). Note that $0 < \alpha(x) < \beta(x), 0 < x < 1$, since a, b are positive on $(0, 1)$.

Remark 3.2. The support of σ is a bounded interval in $(0, +\infty)$, given by $\operatorname{supp}(\sigma) = [\min_{0\leqslant x\leqslant 1} \alpha(x), \max_{0\leqslant x\leqslant 1} \beta(x)]$. Clearly, the $\lambda_{1,N}, \ldots, \lambda_{N,N}$ are then dense in this interval $\operatorname{supp}(\sigma)$ as $N \to \infty$. Moreover, from theorem 2.6 it follows that $\lim_{N\to\infty} \lambda_{1,N} = \min_{0\leqslant x\leqslant 1} \alpha(x)$ and $\lim_{N\to\infty} \lambda_{N,N} = \max_{0\leqslant x\leqslant 1} \beta(x)$.

Remark 3.3. As a consequence of theorem 3.1 and lemma 2.10 we get

$$\begin{aligned} U^{\sigma}(z) &= \int_0^1 U^{\upsilon_{[\alpha(x),\beta(x)]}}(z)\, \mathrm{d}x \\ &= -\frac{1}{2}\int_0^1 \log b(x)\, \mathrm{d}x + \frac{1}{2}\log\frac{1}{|z|} \\ &\quad -\frac{1}{2}\int_0^1 (g_{[\alpha(x),\beta(x)]}(z,\infty) + g_{[\alpha(x),\beta(x)]}(z,0))\, \mathrm{d}x. \end{aligned}$$

Clearly, if $\log b(x) = o(\frac{1}{x(1-x)})$ for $x \downarrow 0$ and $x \uparrow 1$, then U^{σ} is continuous in $\mathbb{C}$.

Next we assume that the limit in (3.2) exists. As a consequence of remark 3.2, limit (3.2) indeed defines a function on the bounded interval $\mathrm{supp}(\sigma)$, clearly satisfying the continuity property. Our goal is then to express R in terms of the functions a and b. We can do this for a rather restricted class of operator data a, b. Here we use a link between the asymptotic zero distributions of the families of Laurent orthogonal polynomials $\{P_{n,N}\}_{n,N\in\mathbb{N}}$, defined by recurrence (3.3), and a constrained minimal energy problem in an external field. This directly follows from a theorem of Dragnev and Saff [11, theorem 3.3], where we need the properties in remarks 3.2 and 3.3.

Lemma 3.4. *Suppose that* $\log(b(x)) = o\big(\frac{1}{x(1-x)}\big)$ *for* $x \downarrow 0$ *and* $x \uparrow 1$ *and that the corresponding sets* $\{\lambda_{j,N}\}_{j=1}^N$, $\{w_{j,N}\}_{j=1}^N$, $N \in \mathbb{N}$, *satisfy the separation condition*

$$\exists\, \rho > 0 \text{ such that } \min_{1\leqslant j\leqslant N-1}(\lambda_{j+1,N} - \lambda_{j,N}) > \frac{\rho}{N} \quad \forall N \in \mathbb{N} \tag{3.6}$$

and the limit relation (3.2). Furthermore, let σ *be defined by (3.1) and* τ_x, $0 < x < 1$, *be the unique measure in* $\mathcal{M}_x^\sigma := \{\mu : \mu(\mathbb{R}) = x \text{ and } 0 \leqslant \mu \leqslant \sigma\}$, *minimizing the logarithmic energy in the external field* $-\frac{1}{2}(xU^{\delta_0} + R)$:

$$J_R(\tau_x; x) = \min_{\mu(\mathbb{R})=x,\mu\leqslant\sigma} J_R(\mu; x), \tag{3.7}$$

with

$$J_R(\mu; x) := \iint \log\frac{1}{|y-y'|}\,\mathrm{d}\mu(y)\,\mathrm{d}\mu(y') + \int (x\log y - R(y))\,\mathrm{d}\mu(y).$$

For the families of Laurent orthogonal polynomials $\{P_{n,N}\}_{n,N\in\mathbb{N}}$, *defined by (3.3), we then have*

$$\lim_{n/N\to x}\frac{1}{n}\sum_{P_{n,N}(z)=0}\delta_z = \frac{\tau_x}{x}, \qquad 0 < x < 1, \tag{3.8}$$

in the sense of weak-⋆ convergence.

Remark 3.5. Since R is continuous on $\mathrm{supp}(\sigma) \subset (0, +\infty)$, each of these extremal measures τ_x, $0 < x < 1$, can also be characterized as the unique measure in $\mathcal{M}_x^\sigma$, satisfying the variational conditions [11, 21]

$$U^{\tau_x}(\lambda) + \frac{x}{2}\log\lambda - \frac{1}{2}R(\lambda)\begin{cases}\geqslant w(\tau_x) & \lambda \in \mathrm{supp}(\sigma - \tau_x)\\ \leqslant w(\tau_x) & \lambda \in \mathrm{supp}(\tau_x)\end{cases} \tag{3.9}$$

for some constant $w(\tau_x) \in \mathbb{R}$.

It is possible to weaken condition (3.6), as shown in [2, 11]. However, this is not the most relevant issue in our analysis. Before stating a theorem about the function R, we discuss some of its properties.

Lemma 3.6. *If the limit in (3.2) exists, then the maximum of* $R : \mathrm{supp}(\sigma) \to (-\infty, 0]$ *is 0.*

Proof. Since $0 < w_{j,N} \leqslant 1$, $1 \leqslant j \leqslant N$, the function $-R$ is non-negative. Further, by the discrete version of Laplace's asymptotic formula, (3.2) and the fact that $\sum_{j=1}^N w_{j,N} = 1$ we obtain

$$\max_{\lambda\in\mathrm{supp}(\sigma)} R(\lambda) = \lim_{N\to\infty}\frac{1}{N}\log\sum_{j=1}^N \mathrm{e}^{NR(\lambda_{j,N})} = \lim_{N\to\infty}\frac{1}{N}\log\sum_{j=1}^N w_{j,N} = 0.$$

□

In analogy with [21, theorem 9.2] we can now prove the following theorem. The extra restriction (iii) on the class of operator data is basically due to Deift and McLaughlin [10].

Theorem 3.7. *Suppose that the operator data* $a, b \in C[0, 1]$ *(and* α, β *constructed by (1.11)), with* $a(x) > 0, 0 \leqslant x \leqslant 1$*, and* $b(x) > 0, 0 < x < 1, b(0) = b(1) = 0$*, satisfy the following conditions:*

(i) $\log b(x) = o(\frac{1}{x(1-x)})$ *for* $x \downarrow 0$ *and* $x \uparrow 1$.

(ii) The separation condition (3.6) and the limit relation (3.2) for the corresponding sets $\{\lambda_{j,N}\}_{j=1}^N, \{w_{j,N}\}_{j=1}^N, N \in \mathbb{N}$*, hold.*

(iii) For each $\lambda \in \operatorname{supp}(\sigma)$ *the set* $\{x \in [0, 1] : \lambda \in [\alpha(x), \beta(x)]\}$ *is an interval, which we denote by*

$$[x_-(\lambda), x_+(\lambda)] := \{x \in [0, 1] : \lambda \in [\alpha(x), \beta(x)]\}. \tag{3.10}$$

Then

$$R(\lambda) = -\int_0^{x_-(\lambda)} (g_{[\alpha(u),\beta(u)]}(\lambda, \infty) + g_{[\alpha(u),\beta(u)]}(\lambda, 0))\,\mathrm{d}u, \qquad \lambda \in \operatorname{supp}(\sigma), \tag{3.11}$$

where, for some $0 < \alpha < \beta$, $g_{[\alpha,\beta]}(z, \infty)$ *and* $g_{[\alpha,\beta]}(z, 0)$ *are the Green functions of the complement of* $[\alpha, \beta]$ *with pole at* ∞ *and* 0*, respectively. Recall that* $0 < \alpha(x) < \beta(x), 0 < x < 1$*, since* a, b *are both positive on* $(0, 1)$.

Remark 3.8. The Green functions $g_{[\alpha,\beta]}(z, \infty)$ and $g_{[\alpha,\beta]}(z, 0)$, with $0 < \alpha < \beta$, are equal to 0 on $[\alpha, \beta]$. Elsewhere they are positive and, as mentioned in lemma 2.10, have the expressions

$$g_{[\alpha,\beta]}(z, \infty) = \log \left| \frac{2z - (\alpha + \beta) + 2\sqrt{(z - \alpha)(z - \beta)}}{\beta - \alpha} \right|, \qquad z \in \mathbb{C}\backslash[\alpha, \beta],$$

$$g_{[\alpha,\beta]}(z, 0) = \log \left| \frac{(\alpha + \beta)z - 2\alpha\beta + 2\sqrt{\alpha\beta}\sqrt{(z - \alpha)(z - \beta)}}{z(\beta - \alpha)} \right|, \qquad z \in \mathbb{C}\backslash[\alpha, \beta],$$

where the square root is such that $\sqrt{(z - \alpha)(z - \beta)}$ is an analytic function of z in $\mathbb{C}\backslash[\alpha, \beta]$, which is positive for $z > \beta$.

Proof. From lemma 3.4 and theorem 2.3 we get that

$$\frac{\tau_x}{x} = \lim_{n/N \to x} \frac{1}{n} \sum_{P_{n,N}(z)=0} \delta_z = \frac{1}{x} \int_0^x \upsilon_{[\alpha(u),\beta(u)]}\,\mathrm{d}u, \qquad 0 < x < 1, \tag{3.12}$$

with $\upsilon_{[\alpha,\beta]}, 0 < \alpha < \beta$, as in (3.5). From (3.4) and condition (iii) it then follows that

$$\operatorname{supp}(\sigma - \tau_x) \cap \operatorname{supp}(\tau_x) = [\alpha(x), \beta(x)], \qquad 0 < x < 1,$$

which changes continuously and is different from a singleton since $b(x) > 0$, for $0 < x < 1$. In these regions we have equality in (3.9). Clearly, knowing the $\tau_x, 0 < x < 1$, the function R is then the unique function, up to a constant, for which there exists constants $w(\tau_x) \in \mathbb{R}, 0 < x < 1$, such that

$$U^{\tau_x}(\lambda) + \frac{x}{2}\log\lambda - \frac{1}{2}R(\lambda) \begin{cases} \geqslant w(\tau_x), & \lambda \in \operatorname{supp}(\sigma - \tau_x), \\ \leqslant w(\tau_x), & \lambda \in \operatorname{supp}(\tau_x). \end{cases}$$

From (3.12) and (2.22) we get, for $\lambda \in \operatorname{supp}(\sigma)$,

$$\begin{aligned} U^{\tau_x}(\lambda) + \frac{x}{2}\log\lambda + \frac{1}{2}\int_0^{x_-(\lambda)} (g_{[\alpha(u),\beta(u)]}(\lambda, \infty) + g_{[\alpha(u),\beta(u)]}(\lambda, 0))\,\mathrm{d}u \\ = -\frac{1}{2}\int_0^x \log b(u)\,\mathrm{d}u - \frac{1}{2}\int_{x_-(\lambda)}^x (g_{[\alpha(u),\beta(u)]}(\lambda, \infty) + g_{[\alpha(u),\beta(u)]}(\lambda, 0))\,\mathrm{d}u, \end{aligned} \tag{3.13}$$

where the first integral on the right-hand side is finite because of condition (i). Note that, by (3.10) and the properties of the Green functions (see remark 3.2),

$$\int_{x_-(\lambda)}^{x} (g_{[\alpha(u),\beta(u)]}(\lambda,\infty) + g_{[\alpha(u),\beta(u)]}(\lambda,0))\,\mathrm{d}u$$
$$= \int_{x_+(\lambda)}^{x} (g_{[\alpha(u),\beta(u)]}(\lambda,\infty) + g_{[\alpha(u),\beta(u)]}(\lambda,0))\,\mathrm{d}u.$$

Furthermore, we observe that by condition (iii)

$$\lambda \in \operatorname{supp}(\tau_x) = \bigcup_{0\leqslant u\leqslant x} [\alpha(u),\beta(u)] \quad\Rightarrow\quad x_-(\lambda) \leqslant x$$

and

$$\lambda \in \operatorname{supp}(\sigma - \tau_x) = \bigcup_{x\leqslant u\leqslant 1} [\alpha(u),\beta(u)] \quad\Rightarrow\quad x \leqslant x_+(\lambda).$$

Since, by definition, the Green functions are non-negative, we then obtain that there exists a constant $C \in \mathbb{R}$ such that $w(\tau_x) = -\frac{1}{2}\int_0^x \log b(u)\,\mathrm{d}u + C$ and

$$R(\lambda) = -\int_0^{x_-(\lambda)} (g_{[\alpha(u),\beta(u)]}(\lambda,\infty) + g_{[\alpha(u),\beta(u)]}(\lambda,0))\,\mathrm{d}u + C.$$

Finally, note that the maximum on the right-hand side in (3.11) is 0, implying $C = 0$ by lemma 3.6. □

Remark 3.9. For operator data satisfying the conditions (i) and (iii) a method that possibly helps to show that the sets $\{w_{\ell,N}\}_{\ell=1}^N$ indeed satisfy a limit relation of the kind (3.2) is the Wentzel–Kramers–Brillouin (WKB) method. This was done for the continuum limit of the Toda lattice in [10].

4. Evolution of the spectral data

In this section we study the evolution of the spectral data for the continuum limit. Suppose we have initial operator data $a(x,0), b(x,0) \in C[0,1]$, with $a(x,0) > 0, 0 \leqslant x \leqslant 1$, and $b(x,0) > 0, 0 < x < 1, b(0,0) = b(1,0) = 0$. We then define $\sigma(0)$ and $R(\cdot,0)$ by the transformation explained in section 3.1. Here we assume the limit (3.2) exists. Now take, for each $N \in \mathbb{N}$,

$$a_{k,N}(0) = a(k/N,0) > 0, \qquad 1 \leqslant k \leqslant N,$$
$$b_{k,N}(0) = b(k/N,0) > 0, \qquad 1 \leqslant k \leqslant N-1,$$

as initial discrete operator data for the discrete finite RTL (1.1). The evolution of the corresponding discrete spectral data (1.5), with the substitution $t \to Nt$, then gives the evolution $\sigma(t)$ and $R(\cdot,t)$.

Theorem 4.1. *If the limits (3.1) and (3.2) hold for $t = 0$, then using evolution (1.5), with $t \to Nt$, they exist for each $t > 0$. In particular, for $t > 0$,*

$$\sigma(t) = \sigma(0) = \sigma, \tag{4.1}$$

and

$$R(\lambda,t) = -\lambda t + R(\lambda,0) - \max_{\lambda\in\operatorname{supp}(\sigma)} (-\lambda t + R(\lambda,0)), \quad \lambda \in \operatorname{supp}(\sigma). \tag{4.2}$$

Proof. As mentioned in the introduction, the $\lambda_{j,N}$, $1 \leqslant j \leqslant N$ are time independent, implying (4.1). Next, from (1.5) we obtain

$$\frac{1}{N}\log w_{j,N}(Nt) = -\lambda_{j,N}t + \frac{1}{N}\log w_{j,N}(0) - \frac{1}{N}\log\left(\sum_{k=1}^{N} w_{k,N}(0)\mathrm{e}^{-\lambda_{k,N}Nt}\right). \tag{4.3}$$

Note that, by the definition of $R(\cdot, 0)$ and the discrete version of Laplace's asymptotic formula,

$$\lim_{N\to\infty}\frac{1}{N}\log\left(\sum_{k=1}^{N}\exp\left[N\left(-\lambda_{k,N}t + \frac{1}{N}\log w_{k,N}(0)\right)\right]\right) = \max_{\lambda\in\mathrm{supp}(\sigma)}(-\lambda t + R(\lambda, 0)).$$

So, letting $N \to \infty$ and taking $j(N)$ such that $\lambda_{j,N} \to \lambda$ as $N \to \infty$, from (4.3) we finally get (4.2). □

5. Inverse problem for the continuum limit of the RTL

In the inverse problem we start from sets of discrete spectral data $0 < \lambda_{1,N} < \cdots < \lambda_{N,N}$ and $w_{j,N} > 0$, $\sum_{j=1}^{N} w_{j,N} = 1$, $N \in \mathbb{N}$, with the following properties. First, suppose that the $\lambda_{j,N}$ satisfy the separation condition (3.6) and that there exists a probability measure σ, for which $\mathrm{supp}(\sigma)$ is a bounded interval in $(0, +\infty)$ and U^σ is continuous on $\mathbb{C}$, such that

$$\lim_{N\to\infty}\lambda_{1,N} = \min(\mathrm{supp}(\sigma)), \qquad \lim_{N\to\infty}\lambda_{N,N} = \max(\mathrm{supp}(\sigma))$$

and limit (3.1) holds. Secondly, assume there exists a continuous function $R : \mathrm{supp}(\sigma) \to (-\infty, 0]$ satisfying (3.2).

Remark 5.1. Note that if these conditions hold for $t = 0$ then they hold for each $t > 0$ following evolution (1.5) of the discrete finite RTL with $t \to Nt$ (see theorem 4.1).

By the inverse spectral transform for the discrete finite RTL, described in section 1.1, we obtain corresponding sets of discrete operator data $a_{n,N} > 0$, $1 \leqslant n \leqslant N$, and $b_{n,N} > 0$, $1 \leqslant n \leqslant N-1$, $N \in \mathbb{N}$. The following theorem shows that if there exist continuous functions $a, b : (0, 1) \to (0, +\infty)$ such that

$$\lim_{n/N\to x} a_{n,N} = a(x), \qquad \lim_{n/N\to x} b_{n,N} = b(x), \qquad 0 < x < 1, \tag{5.1}$$

(under some conditions) they can be obtained by solving the equilibrium problem (3.7).

Theorem 5.2. *Suppose we have discrete spectral data satisfying the properties mentioned above and that for the corresponding discrete operator data limits (5.1) exist. Furthermore, let* τ_x, $0 < x < 1$, *the unique measure in* $\mathcal{M}_x^\sigma$ *minimizing the logarithmic energy in the external field* $-\frac{1}{2}(xU^{\delta_0} + R)$, *see (3.7). If the sets* $\Sigma(\tau_x) := \mathrm{supp}(\sigma - \tau_x) \cap \mathrm{supp}(\tau_x)$, $0 < x < 1$ *are all intervals different from a singleton, then*

$$\Sigma(\tau_x) = [\alpha(x), \beta(x)], \qquad 0 < x < 1, \tag{5.2}$$

where the functions α, β *are constructed from a and b by transformation (1.11).*

Proof. Let $\{P_{n,N}\}_{n=1}^{N}$ be the Laurent orthogonal polynomials with respect to the discrete measure $\sum_{j=1}^{N} w_{j,N}\delta_{\lambda_{j,N}}$, for each $N \in \mathbb{N}$. By the properties assumed on the discrete spectral data we can apply [11, theorem 3.3] in order to obtain

$$\lim_{n/N\to x}\frac{1}{n}\sum_{P_{n,N}(z)=0}\delta_z = \frac{\tau_x}{x}, \qquad 0 < x < 1. \tag{5.3}$$

Then, define $\gamma, \delta : (0, 1) \to \mathrm{supp}(\sigma) \subset (0, +\infty)$ by

$$\Sigma(\tau_x) = [\gamma(x), \delta(x)], \qquad 0 < x < 1.$$

Because of theorem A.10 (a), see the appendix, for each $\lambda \in \mathrm{supp}(\sigma)$ the set $\{x \in (0, 1) : \lambda \in [\gamma(x), \delta(x)]\}$ is an interval. This implies that for γ and δ the interval (0, 1) can be split in two intervals where the function is monotone. So, γ and δ only have a countable number of discontinuity points and the conditions of theorem A.10 (b) with $c = \frac{1}{2}$ and $\nu = \delta_0$ are fulfilled. Together with (5.3) this gives

$$\lim_{n/N\to x} \frac{1}{n} \log \frac{1}{|P_{n,N}(z)|} = \frac{1}{x} \int_0^x U^{\upsilon_{[\gamma(u),\delta(u)]}}(z)\, \mathrm{d}u, \qquad z \in \mathbb{C}\backslash\mathrm{supp}(\tau_x), \qquad 0 < x < 1. \tag{5.4}$$

Since we assumed that limits (5.1) exist, we also get from theorem 2.3 that $\mathrm{supp}(\sigma) = [\min_{0\leqslant x\leqslant 1} \alpha(x), \max_{0\leqslant x\leqslant 1} \beta(x)]$ and

$$\lim_{n/N\to x} \frac{1}{n} \log \frac{1}{|P_{n,N}(z)|} = \frac{1}{x} \int_0^x U^{\upsilon_{[\alpha(u),\beta(u)]}}(z)\, \mathrm{d}u, \qquad z \in \mathbb{C}\backslash[\min_{0\leqslant u\leqslant x} \alpha(u), \max_{0\leqslant u\leqslant x} \beta(u)], \tag{5.5}$$

for each $0 < x < 1$. Combining (5.4) and (5.5) we then obtain, for each fixed $z \in \mathbb{C}\backslash\mathrm{supp}(\sigma)$,

$$U^{\upsilon_{[\gamma(x),\delta(x)]}}(z) = U^{\upsilon_{[\alpha(x),\beta(x)]}}(z), \qquad x \in (0, 1) \quad \text{a.e..}$$

By the unicity theorem of potentials, see, e.g., [32, chapter II, corollary 2.2], this implies that $\upsilon_{[\gamma(x),\delta(x)]} = \upsilon_{[\alpha(x),\beta(x)]}$, and so $[\gamma(x), \delta(x)] = [\alpha(x), \beta(x)]$, for almost every $x \in (0, 1)$. Note that α, β are continuous and that by theorem A.10 (a) the functions γ, δ are either left- or right-continuous in each of their discontinuity points. The equality then holds for all $x \in (0, 1)$ which proves (5.2). □

Remark 5.3. Since the measures τ_x are increasing in $x \in (0, 1)$, the operator data a and b we find in this way clearly satisfy condition (iii) of theorem 3.2.

From the equilibrium problem (3.7) we get nth root asymptotics for the polynomials $P_{n,N}$ (defined as in the proof of theorem 5.2). To remove the assumption that limits (5.1) exist, we would need a stronger asymptotic formula like that for the ratio $P_{n-1,N}/P_{n,N}$. As in the case of discrete orthogonal polynomials, for discrete Laurent orthogonal polynomials this is still an open problem.

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Appendix. Buyarov–Rakhmanov–type formula for families of extremal measures in an external field linearly depending on the mass of the measure and with a constraint on $\mathbb{R}$

A.1. Introduction

In this appendix we generalize a theorem of V S Buyarov and E A Rakhmanov concerning families of extremal measures, see [6]. In particular we consider measures constrained by a probability measure on the real axis and, secondly, allow that the external field linearly depends on the mass of the corresponding extremal measure.

First of all we introduce some notation and assumptions which will be fixed throughout this appendix. Denote by $\mathcal{M}_1(\mathbb{R})$ the set of Borel probability measures on $\mathbb{R}$ and let σ be a *constraint*, satisfying

(i) $\sigma \in \mathcal{M}_1(\mathbb{R})$ with compact support;
(ii) $U^\sigma(z) = -\int \log|z-y|\,\mathrm{d}\sigma(y)$ is continuous in $\mathbb{C}$.

Note that condition (ii) implies that σ has finite logarithmic energy $I(\sigma) = \int U^\sigma \mathrm{d}\sigma$. Next, let $Q : \mathrm{supp}(\sigma) \to (-\infty, +\infty)$ be continuous, and hence an *admissible external field*. Finally, let $Q^\nu = -cU^\nu$, where $0 \leqslant c < 1$ and ν is a probability measure of compact support disjoint from $\mathrm{supp}(\sigma)$, both fixed throughout this appendix. Note that $xQ^\nu + Q$ is then a continuous function on $\mathrm{supp}(\sigma)$ for each $x \in (0, 1)$.

We now consider for each $x \in (0, 1)$ the problem of minimizing the logarithmic energy in the external field $xQ^\nu + Q$, which is

$$J_{xQ^\nu+Q}(\mu) = I(\mu) + 2\int (xQ^\nu + Q)\,\mathrm{d}\mu \in (-\infty, +\infty], \tag{A.1}$$

among all measures in $\mathcal{M}_x^\sigma := \{\mu : \mu(\mathbb{R}) = x \text{ and } 0 \leqslant \mu \leqslant \sigma\}$. Here by $\mu \leqslant \sigma$ we mean that $\sigma - \mu$ is a positive Borel measure. In this context we then get from [11, theorem 2.1] the following results.

Theorem A.4. *Suppose σ, Q and Q^ν satisfy the assumptions mentioned above. For each $x \in (0, 1)$ there exists a unique measure $\tau_x \in \mathcal{M}_x^\sigma$ such that*

$$J_{xQ^\nu+Q}(\tau_x) = \min_{\mu\in\mathcal{M}_x^\sigma} J_{xQ^\nu+Q}(\mu) < +\infty. \tag{A.2}$$

Moreover, if we define for each $\mu \in \mathcal{M}_x^\sigma$ the constant

$$w(\mu) = \min_{\lambda\in\mathrm{supp}(\sigma-\mu)} U^\mu(\lambda) + xQ^\nu(\lambda) + Q(\lambda), \tag{A.3}$$

then τ_x is also the unique measure in $\mathcal{M}_x^\sigma$, satisfying the variational conditions

$$U^{\tau_x}(\lambda) + xQ^\nu(\lambda) + Q(\lambda) \begin{cases} \geqslant w(\tau_x), & \lambda \in \mathrm{supp}(\sigma - \tau_x), \\ \leqslant w(\tau_x), & \lambda \in \mathrm{supp}(\tau_x), \end{cases} \tag{A.4}$$

or the maximization problem $w(\tau_x) = \max_{\mu\in\mathcal{M}_x^\sigma} w(\mu)$.

Remark A.5. For any measure $\mu \in \mathcal{M}_x^\sigma$ we can write $U^\mu = U^\sigma - U^{\sigma-\mu}$. We know that U^μ and $U^{\sigma-\mu}$ are lower semi-continuous, so condition (ii) then implies that U^μ is continuous in $\mathbb{C}$.

Remark A.6. By remark A.5, for any $\mu \in \mathcal{M}_x^\sigma$ we have that $U^\mu + xQ^\nu + Q$ is continuous on $\mathrm{supp}(\sigma)$. This implies that we can indeed drop the 'a.e.' in the first inequality of (A.4).

Remark A.7 (Dual problem)**.** It is easily verified that $\sigma - \tau_x$ is the solution of minimizing the logarithmic energy in the continuous external field $(1-x)Q^\nu - (Q + Q^\nu + U^\sigma)$ among all measures in $\mathcal{M}_{1-x}^\sigma$, see [11, corollary 2.10].

Before stating our main result concerning this minimization problem we need to introduce some other concepts. For each compact set $E \subseteq \mathrm{supp}(\sigma)$ of positive capacity we define υ_E as the corresponding extremal measure associated with the external field $Q^\nu = -cU^\nu$, $0 \leqslant c < 1$. (Here c and ν are fixed throughout this appendix.) This means that υ_E is the unique element in $\mathcal{M}_1(E)$ so that

$$J_{Q^\nu}(\upsilon_E) = \min_{\mu\in\mathcal{M}_1(E)} J_{Q^\nu}(\mu) = \min_{\mu\in\mathcal{M}_1(E)} I(\mu) + 2\int Q^\nu\,\mathrm{d}\mu.$$

As an easy consequence of the properties of balayage measures (see, e.g., [32, chapter II, example 4.8]), we get that $\upsilon_E = c\hat{\upsilon}_E + (1-c)\omega_E$, where $\hat{\upsilon}_E$ is the balayage of ν onto E and ω_E is the classical equilibrium distribution of the compact set E. Indeed, U^{ω_E} is constant on E quasi-everywhere and for some constant C we have $U^{\hat{\upsilon}_E} = U^{\nu} + C$ quasi-everywhere on E. We then get that there exists a constant F_E so that

$$U^{\upsilon_E}(\lambda) + Q^{\nu}(\lambda) = F_E, \qquad \lambda \in E \quad \text{q.e.} \tag{A.5}$$

which is another way to define υ_E among the measures in $\mathcal{M}_1(E)$ uniquely.

Remark A.8. The function U^{υ_E} is continuous in $\mathbb{C}\backslash\mathrm{supp}(\upsilon_E)$ and continuous quasi-everywhere in $\mathbb{C}$, see [32, chapter I, theorem 4.4].

Remark A.9. Note that $U^{\upsilon_E} + Q^{\nu} = U^{\upsilon_E - c\nu}$ is subharmonic in $\mathbb{C}\backslash E$, continuous quasi-everywhere in a neighbourhood of E (since E is disjoint from $\mathrm{supp}(\nu)$) and bounded from above at infinity. Applying the generalized maximum principle, by (A.5) we then get

$$U^{\upsilon_E}(z) + Q^{\nu}(z) \leqslant F_E, \qquad z \in \mathbb{C}.$$

In this appendix we prove the following theorem.

Theorem A.10. *Let $\sigma \in \mathcal{M}_1(\mathbb{R})$ have compact support and continuous logarithmic potential. Furthermore, let Q be continuous and $Q^{\nu} = -cU^{\nu}$, where $0 \leqslant c < 1$ and ν a probability measure of compact support disjoint from* $\mathrm{supp}(\sigma)$. *Then the following holds for the corresponding extremal measures $\tau_x, 0 < x < 1$, in the external field $xQ^{\nu} + Q$ and with constraint σ.*

(a) The map $x \mapsto \tau_x$ is increasing on $(0, 1)$ which means that, for $\epsilon \in (0, 1-x)$, the difference $\tau_{x+\epsilon} - \tau_x$ is a positive Borel measure. This implies that

$$\lim_{u\to x} \tau_u = \tau_x, \qquad x \in (0,1),$$

in the sense of weak-$\star$ convergence.

(b) Let $\Sigma(\tau_x) := \mathrm{supp}(\tau_x) \cap \mathrm{supp}(\sigma - \tau_x)$, then $\Sigma(\tau_x)$ is compact. Furthermore, define the capacity of an arbitrary Borel set as its inner capacity, see, e.g., [37, chapter I, (1.10)]. Under the condition that the sets

$$N_0 := \{x \in (0,1) | \mathrm{cap}(\Sigma(\tau_x)) = 0\}, \tag{A.6}$$

$$N^{\star} := \{x \in (0,1) | \mathrm{cap}\left(\Sigma(\tau_x) \Big\backslash \bigcup_{\epsilon>0} (\mathrm{supp}(\tau_x) \cap \mathrm{supp}(\sigma - \tau_{x+\epsilon}))\right) \neq 0\}, \tag{A.7}$$

$$N_{\star} := \{x \in (0,1) | \mathrm{cap}\left(\Sigma(\tau_x) \Big\backslash \bigcup_{\epsilon>0} (\mathrm{supp}(\tau_{x-\epsilon}) \cap \mathrm{supp}(\sigma - \tau_x))\right) \neq 0\} \tag{A.8}$$

are countable, the measures τ_x satisfy the representation

$$\tau_x = \int_0^x \upsilon_{\Sigma(\tau_u)}\, \mathrm{d}u. \tag{A.9}$$

Part (a) of this theorem was already proven in [19, proposition 4.1] in the case $c = 0$. The proof of the more general case will be similar. Also note that, for $c = 0$, part (b) was posed as an open problem in [1]. Some progress in this context was obtained in [23, theorem 2.1], [10, chapter 4], [20, lemma 3.1, theorem 3.3] for some special constraints and external fields.

A.2. The proof of theorem A.10

Proof of theorem A.10 (a). In order to simplify some of the expressions below we introduce for each $x \in (0, 1)$ and each $\epsilon \in (0, 1 - x)$ the notation

$$\Sigma(\sigma - \tau_x; \epsilon) := \text{supp}(\tau_{x+\epsilon}) \cap \text{supp}(\sigma - \tau_x), \tag{A.10}$$

$$\Sigma(\tau_x; \epsilon) := \text{supp}(\tau_x) \cap \text{supp}(\sigma - \tau_{x+\epsilon}), \tag{A.11}$$

which are compact sets.

We now study the function $U^{\tau_x} - U^{\tau_{x+\epsilon}} - \epsilon Q^\nu$. Set $\Omega = \mathbb{C}\backslash\Sigma(\sigma - \tau_x; \epsilon)$. It is then easy to see that $(\tau_x - \tau_{x+\epsilon})|_\Omega$ (which denotes the restriction of $\tau_x - \tau_{x+\epsilon}$ to the set Ω) is a positive measure. Since

$$\begin{aligned} U^{\tau_x}(z) - U^{\tau_{x+\epsilon}}(z) - \epsilon Q^\nu(z) &= U^{\tau_x - \tau_{x+\epsilon} + \epsilon c\nu}(z) \\ &= U^{(\tau_x - \tau_{x+\epsilon})|_\Omega}(z) + U^{(\tau_x - \tau_{x+\epsilon})|_{\Sigma(\sigma - \tau_x;\epsilon)}}(z) + \epsilon c U^\nu(z), \end{aligned}$$

this implies that the left-hand side is superharmonic in the domain Ω. Next, note that the total mass of $\tau_x - \tau_{x+\epsilon} + \epsilon c\nu$ (with compact support) is less than or equal to 0, hence $U^{\tau_x} - U^{\tau_{x+\epsilon}} - \epsilon Q^\nu$ is bounded from below at infinity. Finally, from remark A.5 we obtain that this function is continuous in $\mathbb{C}\backslash\text{supp}(\nu)$, where $\text{supp}(\nu)$ is disjoint from $\Sigma(\sigma - \tau_x; \epsilon)$.

By theorem A.4 we get that

$$U^{\tau_x}(\lambda) + xQ^\nu(\lambda) + Q(\lambda) \begin{cases} \geqslant w(\tau_x), & \lambda \in \text{supp}(\sigma - \tau_x), \\ \leqslant w(\tau_x), & \lambda \in \text{supp}(\tau_x), \end{cases} \tag{A.12}$$

and

$$U^{\tau_{x+\epsilon}}(\lambda) + (x+\epsilon)Q^\nu(\lambda) + Q(\lambda) \begin{cases} \geqslant w(\tau_{x+\epsilon}), & \lambda \in \text{supp}(\sigma - \tau_{x+\epsilon}), \\ \leqslant w(\tau_{x+\epsilon}), & \lambda \in \text{supp}(\tau_{x+\epsilon}). \end{cases} \tag{A.13}$$

This implies

$$U^{\tau_x}(\lambda) - U^{\tau_{x+\epsilon}}(\lambda) - \epsilon Q^\nu(\lambda) \geqslant w(\tau_x) - w(\tau_{x+\epsilon}), \qquad \lambda \in \Sigma(\sigma - \tau_x; \epsilon), \tag{A.14}$$

and by the generalized minimum principle for superharmonic functions [32, chapter I, theorem 2.4] this inequality then holds for $\lambda \in \mathbb{C}$. Together with (A.12) and (A.13), we then also establish

$$U^{\tau_x}(\lambda) - U^{\tau_{x+\epsilon}}(\lambda) - \epsilon Q^\nu(\lambda) = w(\tau_x) - w(\tau_{x+\epsilon}), \qquad \lambda \in \Sigma(\tau_x; \epsilon). \tag{A.15}$$

All this implies $\Sigma(\tau_x; \epsilon) \subseteq \text{supp}((\tau_x - \tau_{x+\epsilon} + \epsilon c\nu)_-)$ or, equivalently, $(\tau_x - \tau_{x+\epsilon} + \epsilon c\nu)|_{\Sigma(\tau_x;\epsilon)} \leqslant 0$ by the fact that a superharmonic function attains its minimum at the boundary (or see [32, chapter IV, theorem 4.5]). This means that $\tau_x|_{\Sigma(\tau_x;\epsilon)} \leqslant \tau_{x+\epsilon}|_{\Sigma(\tau_x;\epsilon)}$, since $\text{supp}(\nu)$ is disjoint from $\Sigma(\tau_x; \epsilon)$. On the complement of $\Sigma(\tau_x; \epsilon)$ we have either $\tau_x = 0$ or $\tau_{x+\epsilon} = \sigma$ which then completes the proof of the first part of theorem A.10. □

To prove the second part of theorem A.10 we need some technical lemmas.

Lemma A.11. *If $E_1 \subseteq E_2$ are both compact subsets of the real axis, then*

$$\text{cap}(E_1) = \text{cap}(E_2) \quad \Leftrightarrow \quad \text{cap}(E_2 \backslash E_1) = 0. \tag{A.16}$$

Proof. Note that this is evident if $\text{cap}(E_1) = 0$. Let $\text{cap}(E_1) = \text{cap}(E_2) > 0$, implying $\omega_{E_1} = \omega_{E_2}$. Since E_1 is a subset of $\mathbb{R}$, from the maximum principle for harmonic functions we then have that $U^{\omega_{E_2}}(z) < -\log(\text{cap}(E_2))$ for $z \in \mathbb{C}\backslash E_1$. Note that $U^{\omega_{E_2}}$ is constant quasi-everywhere on E_2, so we certainly have $\text{cap}(E_2 \backslash E_1) = 0$.

The other implication easily follows from the definition of capacity. □

A second lemma deals with the convergence of potentials.

Lemma A.12. *Let μ_n be a sequence of positive measures with* $\mathrm{supp}(\mu_n) \subseteq K$ *and K compact. If* $\lim_{n\to\infty} \mu_n = \mu$ *in the sense of weak-$\star$ convergence (where μ has finite mass), then* $\lim_{n\to\infty} U^{\mu_n} = U^{\mu}$ *uniformly on compact subsets of* $\mathbb{C}\backslash K$.

Proof. Let $z \in \mathbb{C}\backslash K$ and consider an arbitrary sequence $z_n \to z$. We easily get that

$$U^{\mu_n}(z_n) = -\int \log\left|\frac{z_n - \zeta}{z - \zeta}\right| \mathrm{d}\mu_n(\zeta) + U^{\mu_n}(z). \tag{A.17}$$

Note that there exists a constant $C > 0$ such that $|z - \zeta| \geqslant C$ for each $\zeta \in K$. Furthermore, without loss of generality we can assume that $|z - z_n| < C$ for each $n \in \mathbb{N}$. This immediately gives

$$\log\left|\frac{z_n - \zeta}{z - \zeta}\right| \leqslant \log\left(1 + \left|\frac{z_n - z}{z - \zeta}\right|\right) \leqslant \frac{|z_n - z|}{C}$$

and

$$\log\left|\frac{z_n - \zeta}{z - \zeta}\right| \geqslant \log\left(1 - \left|\frac{z_n - z}{z - \zeta}\right|\right) \geqslant \log\left(1 - \frac{|z_n - z|}{C}\right),$$

implying that the first term in the right-hand side of (A.17) tends to 0. Then note that $-\log|z - \zeta|$ is a continuous function on K, so we finally get $\lim_{n\to\infty} U^{\mu_n}(z_n) = U^{\mu}(z)$. This then proves the lemma. □

Finally, recall that for a compact set $E \subseteq \mathrm{supp}(\sigma)$ of positive capacity we have $\upsilon_E = c\hat{\upsilon}_E + (1 - c)\omega_E$. From [32, (4.2), p 108] and [32, chapter II, theorem 4.7] we can then easily deduce that

$$F_E = c\int g_E(\zeta, \infty)\,\mathrm{d}\nu(\zeta) + (1 - c)\log\frac{1}{\mathrm{cap}(E)}, \tag{A.18}$$

where $g_E(z, \infty)$ is the Green function for the complement of E with pole at ∞. Using the notation (A.6), (A.7), (A.10) and (A.11) we can now prove the following.

Lemma A.13. *For every $x \in (0, 1)\backslash(N_0 \cup N^\star)$ we have*

(a) $\lim_{\epsilon\downarrow 0} \upsilon_{\Sigma(\tau_x;\epsilon)} = \upsilon_{\Sigma(\tau_x)}$ *in the sense of weak-$\star$ convergence,*

(b) $\lim_{\epsilon\downarrow 0} F_{\Sigma(\tau_x;\epsilon)} = F_{\Sigma(\tau_x)}$.

Secondly, for every $x \in (0, 1)\backslash N_0$ for which

$$\mathrm{cap}\left(\bigcap_{\epsilon>0} \Sigma(\sigma - \tau_x; \epsilon)\backslash\Sigma(\tau_x)\right) = 0 \tag{A.19}$$

we have

(a′) $\lim_{\epsilon\downarrow 0} \upsilon_{\Sigma(\sigma-\tau_x;\epsilon)} = \upsilon_{\Sigma(\tau_x)}$ *in the sense of weak-$\star$ convergence,*

(b′) $\lim_{\epsilon\downarrow 0} F_{\Sigma(\sigma-\tau_x;\epsilon)} = F_{\Sigma(\tau_x)}$.

Proof. First of all we fix $x \in (0, 1)\backslash(N_0 \cup N^\star)$. From part (a) of theorem A.10 we see that $\Sigma(\tau_x; \epsilon)$ is an increasing family of compact sets if $\epsilon \downarrow 0$. By [25, lemma 2.10] we then get $\mathrm{cap}\left(\bigcup_{\epsilon>0} \Sigma(\tau_x; \epsilon)\right) = \lim_{\epsilon\downarrow 0} \mathrm{cap}(\Sigma(\tau_x; \epsilon))$, implying $\mathrm{cap}(\Sigma(\tau_x; \epsilon)) > 0$ for very small $\epsilon > 0$. We now only look at such ϵ. Note that all the measures $\upsilon_{\Sigma(\tau_x;\epsilon)}$ are supported on $\Sigma(\tau_x)$, which is compact. So, for each sequence of ϵ-values converging to zero, the corresponding sequence of measures $\upsilon_{\Sigma(\tau_x;\epsilon)}$ has a converging subsequence in the sense of weak-$\star$ convergence. Suppose now that $\epsilon_n \to 0$ and $\lim_{n\to\infty} \upsilon_{\Sigma(\tau_x;\epsilon_n)} = \mu$, where μ is a probability measure on $\Sigma(\tau_x)$. Since for each $n \in \mathbb{N}$

$$U^{\upsilon_{\Sigma(\tau_x;\epsilon_n)}}(z) + Q^\nu(z) = F_{\Sigma(\tau_x;\epsilon_n)}, \qquad z \in \Sigma(\tau_x;\epsilon_n) \quad \text{q.e..}$$

and $\operatorname{cap}\left(\Sigma(\tau_x)\setminus \bigcup_{n\in\mathbb{N}} \Sigma(\tau_x;\epsilon_n)\right) = 0$ we have

$$\liminf_{n\to\infty} U^{\upsilon_{\Sigma(\tau_x;\epsilon_n)}}(z) + Q^\nu(z) = \liminf_{n\to\infty} F_{\Sigma(\tau_x;\epsilon_n)}, \qquad z \in \Sigma(\tau_x) \quad \text{q.e..}$$

So, applying the lower envelope theorem [32, chapter I, theorem 6.9] we then obtain

$$U^\mu(z) + Q^\nu(z) = \liminf_{n\to\infty} F_{\Sigma(\tau_x;\epsilon_n)}, \qquad z \in \Sigma(\tau_x) \quad \text{q.e..}$$

This means that $\mu = \upsilon_{\Sigma(\tau_x)}$ from which, by [3, theorem 2.3], we can conclude part (a) of the lemma. We then also established

$$\liminf_{\epsilon\downarrow 0} F_{\Sigma(\tau_x;\epsilon)} = F_{\Sigma(\tau_x)}$$

which gives part (b) since by (A.18) it is clear that $F_{\Sigma(\tau_x;\epsilon)}$ decreases if $\epsilon \downarrow 0$, implying a limit.

Secondly, note that from part (a) of theorem A.10 we see that $\Sigma(\sigma - \tau_x;\epsilon)$ is a decreasing family of compact sets if $\epsilon \downarrow 0$. Now, let $x \in (0,1)\setminus N_0$ such that (A.19) holds. Clearly, $\operatorname{cap}(\Sigma(\sigma - \tau_x;\epsilon)) > 0$ for every $\epsilon > 0$ and all the corresponding measures $\upsilon_{\Sigma(\sigma-\tau_x;\epsilon)}$ are supported on the compact $\operatorname{supp}(\sigma)$. Thus, for each sequence of ϵ-values converging to zero, the corresponding sequence of measures $\upsilon_{\Sigma(\sigma-\tau_x;\epsilon)}$ has a converging subsequence in the sense of weak-$\star$ convergence. Let $\epsilon_n \to 0$ and $\lim_{n\to\infty} \upsilon_{\Sigma(\tau_x;\epsilon_n)} = \mu$, then μ is a probability measure on $E^\star := \bigcap_{n\in\mathbb{N}} \Sigma(\sigma - \tau_x;\epsilon_n)$. By (A.5) we have

$$\liminf_{n\to\infty} U^{\upsilon_{\Sigma(\sigma-\tau_x;\epsilon_n)}}(z) + Q^\nu(z) = \liminf_{n\to\infty} F_{\Sigma(\sigma-\tau_x;\epsilon_n)}, \qquad z \in E^\star \quad \text{q.e.,}$$

and, applying the lower envelope theorem,

$$U^\mu(z) + Q^\nu(z) = \liminf_{n\to\infty} F_{\Sigma(\sigma-\tau_x;\epsilon_n)}, \qquad z \in E^\star \quad \text{q.e..}$$

So, we established that $\mu = \upsilon_{E^\star}$. Since this measure has no mass on sets of capacity zero, by (A.19) also $\mu = \upsilon_{\Sigma(\tau_x)}$, giving part (a′) of the lemma by [3, theorem 2.3]. Furthermore,

$$\liminf_{\epsilon\downarrow 0} F_{\Sigma(\sigma-\tau_x;\epsilon)} = F_{\Sigma(\tau_x)}$$

which gives part (b′) since $F_{\Sigma(\sigma-\tau_x;\epsilon)}$ increases if $\epsilon \downarrow 0$, implying a limit. □

We are now ready to continue the proof of theorem A.10.

Proof of theorem A.10 (b). From part (a) of theorem A.10, $\operatorname{supp}(\tau_x)$ is an increasing family of compact sets. So, the increasing function $\operatorname{cap}(\operatorname{supp}(\tau_x))$ has at most countably many discontinuity points. For a continuity point x we have, since capacity is upper continuous and by lemma A.11, that

$$\operatorname{cap}\left(\bigcap_{\epsilon>0} \operatorname{supp}(\tau_{x+\epsilon})\right) = \operatorname{cap}(\operatorname{supp}(\tau_x)) \quad \text{or} \quad \operatorname{cap}\left(\bigcap_{\epsilon>0} \operatorname{supp}(\tau_{x+\epsilon})\setminus\operatorname{supp}(\tau_x)\right) = 0.$$

This immediately implies that

$$\tilde{N}^\star := \left\{ x \in (0,1) \,\middle|\, \operatorname{cap}\left(\bigcap_{\epsilon>0} \Sigma(\sigma - \tau_x;\epsilon)\setminus\Sigma(\tau_x)\right) \neq 0 \right\}$$

is countable.

Fix $x \in (0,1)\backslash(N_0 \cup N^\star \cup \tilde{N}^\star)$ and $\epsilon > 0$ small enough such that $\mathrm{cap}(\Sigma(\tau_x;\epsilon)) > 0$. We explained already in the beginning of the proof of lemma A.13 that this is possible. Furthermore, we clearly have $\mathrm{cap}(\Sigma(\tau_x;\epsilon)) > 0$. Now, define the two functions

$$G_1(z) := U^{\tau_{x+\epsilon}-\tau_x}(z) - \epsilon U^{\upsilon_{\Sigma(\tau_x;\epsilon)}}(z), \tag{A.20}$$

$$G_2(z) := U^{\tau_x-\tau_{x+\epsilon}}(z) + \epsilon U^{\upsilon_{\Sigma(\sigma-\tau_x;\epsilon)}}(z). \tag{A.21}$$

In part (a) of the theorem we proved that $\tau_{x+\epsilon} - \tau_x$ is a positive measure. So, G_1 is superharmonic in $\mathbb{C}\backslash\Sigma(\tau_x;\epsilon)$, continuous quasi-everywhere in $\mathbb{C}$ (by remarks A.5 and A.8) and bounded at infinity. Recalling (A.15), we also have

$$\begin{aligned} G_1(\lambda) &= U^{\tau_{x+\epsilon}-\tau_x}(\lambda) + \epsilon Q^\nu(\lambda) - \epsilon\,(U^{\upsilon_{\Sigma(\tau_x;\epsilon)}}(\lambda) + Q^\nu(\lambda)) \\ &= w(\tau_{x+\epsilon}) - w(\tau_x) - \epsilon F_{\Sigma(\tau_x;\epsilon)}, \qquad \lambda \in \Sigma(\tau_x;\epsilon) \quad \text{q.e..} \end{aligned}$$

From the generalized minimum principle for superharmonic functions [32, chapter I, theorem 2.4] we then obtain

$$G_1(z) \geqslant w(\tau_{x+\epsilon}) - w(\tau_x) - \epsilon F_{\Sigma(\tau_x;\epsilon)}, \qquad z \in \mathbb{C}. \tag{A.22}$$

Now, by (A.14) we get

$$\begin{aligned} w(\tau_{x+\epsilon}) - w(\tau_x) &\geqslant \int (U^{\tau_{x+\epsilon}-\tau_x} + \epsilon Q^\nu)\,\mathrm{d}\upsilon_{\Sigma(\sigma-\tau_x;\epsilon)} \\ &= \int (U^{\upsilon_{\Sigma(\sigma-\tau_x;\epsilon)}} + Q^\nu)\,\mathrm{d}(\tau_{x+\epsilon} - \tau_x) + c\int (U^{\tau_{x+\epsilon}-\tau_x} - \epsilon U^{\upsilon_{\Sigma(\sigma-\tau_x;\epsilon)}})\,\mathrm{d}\nu, \end{aligned}$$

where the last step is allowed since $\mathrm{supp}(\nu)$ is disjoint from $\mathrm{supp}(\sigma)$. Since $\tau_{x+\epsilon} - \tau_x$ is c-absolutely continuous (which means that they have no mass on zero capacity sets) and supported on $\Sigma(\sigma - \tau_x;\epsilon)$, by (A.5) and (A.22) we then have

$$\begin{aligned} w(\tau_{x+\epsilon}) - w(\tau_x) &\geqslant \epsilon F_{\Sigma(\sigma-\tau_x;\epsilon)} + c\int G_1\,\mathrm{d}\nu + \epsilon c\int (U^{\upsilon_{\Sigma(\tau_x;\epsilon)}} - U^{\upsilon_{\Sigma(\sigma-\tau_x;\epsilon)}})\,\mathrm{d}\nu \\ &\geqslant \epsilon F_{\Sigma(\sigma-\tau_x;\epsilon)} + c\big(w(\tau_{x+\epsilon}) - w(\tau_x) - \epsilon F_{\Sigma(\tau_x;\epsilon)}\big) \\ &\quad + \epsilon c\int (U^{\upsilon_{\Sigma(\tau_x;\epsilon)}} - U^{\upsilon_{\Sigma(\sigma-\tau_x;\epsilon)}})\,\mathrm{d}\nu, \end{aligned}$$

implying

$$w(\tau_{x+\epsilon}) - w(\tau_x) \geqslant \frac{\epsilon\left(F_{\Sigma(\sigma-\tau_x;\epsilon)} - cF_{\Sigma(\tau_x;\epsilon)}\right)}{1-c} + \frac{\epsilon c}{1-c}\int (U^{\upsilon_{\Sigma(\tau_x;\epsilon)}} - U^{\upsilon_{\Sigma(\sigma-\tau_x;\epsilon)}})\,\mathrm{d}\nu.$$

Combining this with (A.22) we then finally conclude that

$$\begin{aligned} \frac{U^{\tau_{x+\epsilon}}(z) - U^{\tau_x}(z)}{\epsilon} &\geqslant \frac{F_{\Sigma(\sigma-\tau_x;\epsilon)} - F_{\Sigma(\tau_x;\epsilon)}}{1-c} \\ &\quad + \frac{c}{1-c}\int (U^{\upsilon_{\Sigma(\tau_x;\epsilon)}} - U^{\upsilon_{\Sigma(\sigma-\tau_x;\epsilon)}})\,\mathrm{d}\nu + U^{\upsilon_{\Sigma(\tau_x;\epsilon)}}(z), \end{aligned} \tag{A.23}$$

for each $z \in \mathbb{C}$.

Secondly, the function G_2 is superharmonic in $\mathbb{C}\backslash\Sigma(\sigma - \tau_x;\epsilon)$ since $\tau_{x+\epsilon} - \tau_x$ is a positive measure on $\Sigma(\sigma - \tau_x;\epsilon)$, continuous quasi-everywhere in $\mathbb{C}$ by remarks A.5 and A.8 and bounded at infinity. From (A.14), we also obtain

$$\begin{aligned} G_2(\lambda) &= U^{\tau_x-\tau_{x+\epsilon}}(\lambda) - \epsilon Q^\nu(\lambda) + \epsilon\,(U^{\upsilon_{\Sigma(\sigma-\tau_x;\epsilon)}}(\lambda) + Q^\nu(\lambda)) \\ &\geqslant w(\tau_x) - w(\tau_{x+\epsilon}) + \epsilon F_{\Sigma(\sigma-\tau_x;\epsilon)}, \qquad \lambda \in \Sigma(\sigma - \tau_x;\epsilon) \quad \text{q.e.} \end{aligned}$$

which then by the generalized minimum principle implies

$$G_2(z) \geqslant w(\tau_x) - w(\tau_{x+\epsilon}) + \epsilon F_{\Sigma(\sigma-\tau_x;\epsilon)}, \qquad z \in \mathbb{C}. \tag{A.24}$$

In a similar way as above, by (A.15) we have

$$\begin{aligned} w(\tau_x) - w(\tau_{x+\epsilon}) &= \int (U^{\tau_x - \tau_{x+\epsilon}} - \epsilon Q^\nu)\, \mathrm{d}\upsilon_{\Sigma(\tau_x;\epsilon)} \\ &= -\int (U^{\upsilon_{\Sigma(\tau_x;\epsilon)}} + Q^\nu)\, \mathrm{d}(\tau_{x+\epsilon} - \tau_x) + c\int (\epsilon U^{\upsilon_{\Sigma(\tau_x;\epsilon)}} - U^{\tau_{x+\epsilon}-\tau_x})\, \mathrm{d}\nu. \end{aligned}$$

By remark A.9 and (A.24) we then establish

$$\begin{aligned} w(\tau_x) - w(\tau_{x+\epsilon}) &\geqslant -\epsilon F_{\Sigma(\tau_x;\epsilon)} + c\int G_2\, \mathrm{d}\nu + \epsilon c\int (U^{\upsilon_{\Sigma(\tau_x;\epsilon)}} - U^{\upsilon_{\Sigma(\sigma-\tau_x;\epsilon)}})\, \mathrm{d}\nu \\ &\geqslant -\epsilon F_{\Sigma(\tau_x;\epsilon)} + c\big(w(\tau_x) - w(\tau_{x+\epsilon}) + \epsilon F_{\Sigma(\sigma-\tau_x;\epsilon)}\big) \\ &\quad + \epsilon c\int (U^{\upsilon_{\Sigma(\tau_x;\epsilon)}} - U^{\upsilon_{\Sigma(\sigma-\tau_x;\epsilon)}})\, \mathrm{d}\nu, \end{aligned}$$

so that

$$w(\tau_x) - w(\tau_{x+\epsilon}) \geqslant \frac{\epsilon\big(cF_{\Sigma(\sigma-\tau_x;\epsilon)} - F_{\Sigma(\tau_x;\epsilon)}\big)}{1-c} + \frac{\epsilon c}{1-c}\int (U^{\upsilon_{\Sigma(\tau_x;\epsilon)}} - U^{\upsilon_{\Sigma(\sigma-\tau_x;\epsilon)}})\, \mathrm{d}\nu.$$

Finally, combine this with (A.24) to find

$$\begin{aligned} \frac{U^{\tau_{x+\epsilon}}(z) - U^{\tau_x}(z)}{\epsilon} &\leqslant \frac{F_{\Sigma(\tau_x;\epsilon)} - F_{\Sigma(\sigma-\tau_x;\epsilon)}}{1-c} \\ &\quad + \frac{c}{1-c}\int (U^{\upsilon_{\Sigma(\sigma-\tau_x;\epsilon)}} - U^{\upsilon_{\Sigma(\tau_x;\epsilon)}})\, \mathrm{d}\nu + U^{\upsilon_{\Sigma(\sigma-\tau_x;\epsilon)}}(z), \end{aligned} \tag{A.25}$$

for each $z \in \mathbb{C}$.

Let $\epsilon \downarrow 0$ in (A.23) and (A.25). By lemmas A.13 and A.12 and since the compact $\operatorname{supp}(\nu)$ is disjoint from the compact $\operatorname{supp}(\sigma)$, we first of all obtain

$$\lim_{\epsilon\downarrow 0} \frac{U^{\tau_{x+\epsilon}}(z) - U^{\tau_x}(z)}{\epsilon} = U^{\upsilon_{\Sigma(\tau_x)}}(z), \qquad z \in \mathbb{C}\backslash\operatorname{supp}(\sigma),$$

for each $x \in (0,1)\backslash(N_0 \cup N^\star \cup \tilde{N}^\star)$, where $N_0 \cup N^\star \cup \tilde{N}^\star$ is countable (by the assumptions of the theorem). As an easy consequence of the fact that $N_\star$ is countable and remark A.7, we also have

$$\lim_{\epsilon\downarrow 0} \frac{U^{\tau_{x-\epsilon}}(z) - U^{\tau_x}(z)}{-\epsilon} = \lim_{\epsilon\downarrow 0} \frac{U^{\sigma-\tau_{x-\epsilon}}(z) - U^{\sigma-\tau_x}(z)}{\epsilon} = U^{\upsilon_{\Sigma(\tau_x)}}(z), \qquad z \in \mathbb{C}\backslash\operatorname{supp}(\sigma),$$

for each $x \in (0,1)$, except for a countable set of points. Thus, there exists a countable set $N \subset (0,1)$ such that, for each $x \in (0,1)\backslash N$, the set $\Sigma(\tau_x)$ has positive capacity and

$$\frac{\mathrm{d}}{\mathrm{d}x} U^{\tau_x}(z) = U^{\upsilon_{\Sigma(\tau_x)}}(z), \qquad z \in \mathbb{C}\backslash\operatorname{supp}(\sigma). \tag{A.26}$$

Let μ_x the Borel measure in $\mathcal{M}_x(\operatorname{supp}(\sigma))$, acting on an arbitrary Borel set E like

$$\mu_x(E) = \int_0^x \upsilon_{\Sigma(\tau_u)}(E)\, \mathrm{d}u,$$

which is well defined since N_0 is countable. Next, let $z \in \mathbb{C}\backslash\operatorname{supp}(\sigma)$ be fixed. Since $-\log|z-y|$ is continuous on $\operatorname{supp}(\sigma)$, by the definition of integration we then get

$$U^{\mu_x}(z) = \int_0^x U^{\upsilon_{\Sigma(\tau_u)}}(z)\, \mathrm{d}u.$$

Note that there exists a constant C such that $|\log|z-y|| \leqslant C$ for each $y \in \text{supp}(\sigma)$. So, since $|U^{\upsilon_{\Sigma(\tau_u)}}(z)| \leqslant C$ for each $u \in (0,1)\backslash N_0$ and by theorem A.10 (a), it is easy to see that $U^{\tau_x}(z) - U^{\mu_x}(z)$ is a continuous function of $x \in (0,1)$. Together with (A.26) we also obtain that $\frac{\mathrm{d}}{\mathrm{d}x}(U^{\tau_x}(z) - U^{\mu_x}(z)) = 0$ for each $x \in (0,1)\backslash N$. Furthermore, since τ_x and μ_x converge to the zero measure in the sense of weak-$\star$ convergence if x tends to zero and $-\log|z-y|$ is continuous on supp(σ),

$$\lim_{x\downarrow 0} U^{\tau_x}(z) - U^{\mu_x}(z) = 0.$$

By [7, theorem 6.3.10], all this implies that, for each fixed $z \in \mathbb{C}\backslash\text{supp}(\sigma)$,

$$U^{\tau_x}(z) - U^{\mu_x}(z) = 0, \qquad x \in (0,1).$$

From the unicity theorem [32, chapter II, corollary 2.2] we then finally establish (A.9) for each $x \in (0,1)$. □

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